

Sources and Sinks of Available Potential Energy in a Moist Atmosphere

Olivier Pauluis¹

Courant Institute of Mathematical Sciences

New York University

Submitted to the *Journal of the Atmospheric Sciences*

April 29th, 2005

¹ Corresponding author address: O. Pauluis, Warren Weaver Hall, 251 Mercer street, New York City, NY 10012-1185. E-mail: pauluis@cims.nyu.edu.

Abstract

A general expression for the sources and sinks of available potential energy (APE) in a moist atmosphere is derived. The available potential energy here is defined as the difference between the total static energy of the atmosphere and that of a reference state that minimizes the total static energy after a sequence of reversible adiabatic transformations. The central result presented here is that small reorganizations of the reference state do not change its total static energy. This makes it possible to determine the rate of change of APE even in the absence of an analytic formula for the reference, as it is the case in a moist atmosphere. The effects of changing the total water content of an air parcel are also considered in order to evaluate the change of APE due to precipitation, evaporation and diffusion of water vapor. These results are then discussed in the context of the APE production for an idealized radiative convective equilibrium.

1. Introduction

The maintenance of the general circulation requires that kinetic energy be continuously generated to balance frictional dissipation. In this respect, the atmosphere behaves as a heat engine that produces mechanical work by transporting heat from warm to cold regions. However, the amount of kinetic energy produced is limited by the thermodynamic properties of moist convection. Pauluis et al. (2000) and Pauluis and Held (2002a,b) have shown that irreversibilities associated with the Earth's hydrological cycle severely reduce the amount of kinetic energy generated by the atmospheric circulation. Understanding the thermodynamic constraints on kinetic energy production is a key element in building a theory of the general circulation of the atmosphere.

Early discussions of the maintenance of the general circulation were based on the notion of Available Potential Energy (APE) rather than entropy. Lorenz (1955) defines the APE as the difference in total static energy (internal plus geopotential) between the current state of the atmosphere and that of an idealized reference state, defined as the state that minimizes the static energy of the atmosphere after a sequence of reversible adiabatic transformations. The APE is thus a property of the entire atmosphere, in contrast to entropy, which is well-defined for individual air parcels. By construction, the sum of APE and total kinetic energy is conserved through reversible adiabatic transformations. The atmospheric circulation can be described as a heat engine in which external heat sources act as a net source of APE, which is then converted into kinetic energy by the mostly reversible, adiabatic motions before being dissipated through frictional dissipation.

It should be noted that this description of the atmospheric circulation in terms of APE has been limited to dry circulations. Moisture, if it is taken into consideration at all, only enters as a prescribed external heat source, accounting for the 'latent heat release' associated with phase transitions of water. From a conceptual point of view, the latent heat release is not an external process, but corresponds to an internal energy conversion, and the actual external heating occurs when water evaporates at the Earth's surface. The main limitation of the dry framework lies in that it requires a priori knowledge of the distribution of latent heat release in order to determine the source of APE. Given the new understanding of the importance of latent heat transport and moist processes for the entropy production, the dry framework seems inadequate to properly address the maintenance of the general circulation in the Earth's atmosphere.

Applications of APE to moist circulations have however been limited. Lorenz (1977) and Randal and Wang (1992) show that the concept of APE, defined as the static energy difference between the current and reference states, can be applied to a moist atmosphere as long as the transformations leading to the reference state include reversible phase transitions. The fundamental difficulty here is that the reference state cannot be derived analytically, but must be determined from an iterative minimization procedure. In the absence of an analytic expression, it becomes impossible to take a time-derivative required to compute the evolution of APE. It should be stressed here that, while a number of textbooks discuss the general circulation in terms of APE, none provides an expression for the sources and sinks of APE in a moist atmosphere. They rely instead on a formulation that is only valid in the context of a dry atmosphere.

The issue of the sources and sinks of APE has received little attention so far, to the exception of the recent studies Bannon (2004, 2005), which derive the effects of moisture in the context of various formulations for the available energy. Bannon (2004) obtains a general formula for the sources and sinks of available energy, but this expression requires a precise knowledge of the evolution of the reference state. The purpose of the current paper is primarily to derive an analytic expression for the rate of change of the Lorenz APE. The main result here is the rate of change of APE associated with changes in the reference pressure cancels out. This makes it possible to obtain an expression for the sources and sinks of APE even in the absence of an analytic formula for the pressure in the reference state. This expression is equally valid for dry and moist atmospheres, and requires only the knowledge of the reference state, not its evolution.

The individual contributions of various processes are analyzed. In particular, an external heat source generates an amount of APE given by the net energy input multiplied by a 'potential efficiency'. This potential efficiency is equal to that of a perfect heat engine acting between the parcel's temperature and its temperature in the reference state. The efficiency for evaporation is slightly lower, mainly due to a water loading contribution. Adiabatic processes can also act as source or sink of APE. It is found that diffusion of sensible and latent heat can result in a reduction of the APE that can be comparable to the surface energy source. In addition, both precipitation and re-evaporation can act as sources of APE. This latter point is in stark contrast with a dry atmosphere where adiabatic processes can only increase the APE through a reverse conversion of kinetic energy.

2. Time tendency of the Available Potential Energy

The total energy in the atmosphere is given by the integral

$$E = \int u + gZ + \frac{1}{2} |V|^2 dm \quad (1)$$

where u is the internal energy, g the gravitational acceleration, Z the geopotential height and V the three-dimensional velocity. The integral in (1) is taken over all the mass of the atmosphere, with dm the mass of an air parcel.

The first law of thermodynamics implies that in the absence of external heat sources and exchanges with the lower boundary, the total energy is conserved. Consider an atmosphere in an initial state A that is transformed, after a set of adiabatic processes, into a final state B . Then the net amount of kinetic energy generated by the successive transformations is equal to the net loss in static energy:

$$\Delta \int \frac{1}{2} |V|^2 dm = -\Delta \int u + gZ dm. \quad (2)$$

Equation (2) is a statement of energy conservation for an isolated system. It does not provide any constraint on how much energy can be converted, which depends on the sequence of transformations considered.

For any atmospheric state, it is possible to define a reference state as the state that minimizes the total static energy after a set of reversible adiabatic transformation. Because of (2), this is also the state that would maximize the kinetic energy in the atmosphere. The Available Potential Energy (APE) is defined as the difference in total static energy between the atmosphere in its current state and in the reference state:

$$APE = \int (u - u_{ref}) + g(Z - Z_{ref}) dm, \quad (3)$$

where u_{ref} and Z_{ref} denote the internal energy and geopotential height of a parcel in its reference state. As entropy s and total water content q_t are conserved in reversible adiabatic transformation, the reference state is a function of the joint distribution of entropy s and humidity q_t . A more detailed description of the properties of the reference state can be found in Lorenz (1977) and Randall and Wang (1992).

A convenient simplification arises when the atmosphere is in hydrostatic balance. Indeed, in this case the vertical integral of the static energy is equal to the vertical integral of the enthalpy. Furthermore, Lorenz (1955) shows that the reference state is always hydrostatic. Therefore, the APE of a hydrostatic atmosphere is given by the difference in the total enthalpy

$$APE = \int h - h_{ref} dm \quad (4)$$

where h and h_{ref} are the enthalpy in the current and reference states.

We are interested here in determining the rate of changes of APE. For simplicity, one can assume that the current state is in hydrostatic balance. In this case, the rate of change of APE is given by the difference between the rate of change of the total enthalpy of the atmosphere and of the reference state:

$$\frac{d}{dt} APE = \int \frac{dh}{dt} dm - \int \frac{dh_{ref}}{dt} dm. \quad (5)$$

In a non-hydrostatic atmosphere, additional contributions arise from the fact that the total static energy is not equal to the total enthalpy, primarily related to elastic processes. For a full discussion of these additional contributions, the reader is referred to Bannon (2004).

In a multiple component system like moist air, it is useful to distinguish between *open* and *closed* transformations, depending on whether a given parcel exchanges mass with its surrounding. Diffusion of heat, radiative heating, evaporation of cloud water, and

expansion and compression are examples of closed transformations, as the total amount of water is conserved. In contrast, evaporation at the Earth's surface, diffusion of water vapor, and precipitation are open transformations, in which the amount of water in an air parcel varies, and require specific treatment.

2.1. Closed transformations.

Consider first the rate of change of APE for closed transformations only. The rate of change of enthalpy is related to the rates of change of entropy, specific humidity and pressure:

$$\begin{aligned}\frac{dh}{dt} &= Q + \alpha \frac{dp}{dt} \\ &= T \frac{ds}{dt} + \Delta\mu \frac{dq}{dt} + \alpha \frac{dp}{dt}.\end{aligned}\tag{6}$$

Here, q is the specific humidity, Q is the external heating rate, $\Delta\mu = \mu_v - \mu_l = R_v \ln H$ is the difference of chemical potential between water vapor and liquid water, with H the relative humidity, and α is the specific volume. The time derivative $\frac{d}{dt}$ here refers to the Lagrangian derivative following a parcel's trajectory. At this point, only closed transformations are considered, so the total water q_T is constant.

Similarly, the enthalpy change in the reference state is given by

$$\frac{dh_{ref}}{dt}(t_0) = T_{ref} \frac{ds}{dt} + \Delta\mu_{ref} \frac{dq}{dt} + \frac{d}{dt} \left(\int_{p_{ref}(t_0)}^{p_{ref}(t_0+t)} \alpha dp \right).\tag{7}$$

As before, the subscript *ref* indicates a quantity evaluated in the reference state. The third term on the right-hand side corresponds to the pressure work as a parcel reference

pressure varies. Note that if p_{ref} is a smooth function of time, this integral is equal to

$\alpha_{ref} \frac{dp_{ref}}{dt}$. The integral notation is kept here, as p_{ref} can be discontinuous.

Hence, combining (6) and (7) into (5) yields

$$\begin{aligned} \frac{d}{dt} APE = & \int (T - T_{ref}) \frac{ds}{dt} dm + \int (\Delta\mu - \Delta\mu_{ref}) \frac{dq}{dt} dm \\ & + \int \alpha \frac{dp}{dt} dm - \int \frac{d}{dt} \left(\int_{p_{ref}(t_0)}^{p_{ref}(t_0+t)} \alpha dp \right) dm. \end{aligned} \quad (8)$$

This equation is similar to that derived by Bannon (2004). The first two terms on the right-hand side include the generation of APE by external heating, but also the effects of diffusion of heat and phase transition. They will be discussed in greater detail in Section

4. The third term on the right-hand side can be written as

$$\begin{aligned} \int \alpha \frac{dp}{dt} dm &= \int \frac{\partial p}{\partial t} dx dy dz + \int \vec{V} \cdot \vec{\nabla} p dx dy dz \\ &= \frac{d}{dt} \int \alpha^{-1} g z dx dy dz - \int p \vec{\nabla} \cdot \vec{V} dx dy dz \\ &= \frac{d\Phi}{dt} - W = -\frac{d}{dt} KE \end{aligned} \quad (9)$$

after replacing the integral over the mass by a volume integral, with $\alpha dm = dx dy dz$.

Equation (9) indicates that the contribution of the total pressure change is equal to

difference between the increase in geopotential energy $\frac{d\Phi}{dt}$ and the work performed by

the atmosphere on itself W . This term thus accounts for the conversion of APE to kinetic

energy $-\frac{d}{dt} KE$. The fourth term of the right-hand side of (8) is the change in APE

associated with a reordering of the reference state. As there is no analytic expression for

p_{ref} , the changes in the reference pressure would have to be obtained numerically. This

would make the computation of the sources and sinks of APE extremely difficult.

However, it is now shown now that the change in APE associated with a reordering of the reference state always vanishes, making the computation of $\frac{dp_{ref}}{dt}$ unnecessary.

Consider two atmospheric reference states A and C described by given distributions of entropy $s_A(m)$ and $s_C(m)$, and water vapor $q_A(m)$ and $q_C(m)$. The pressure distributions in these reference states are $p_A(m)$, and $p_C(m)$, while $h_A(m)$ and $h_C(m)$ are the enthalpy distributions. Here, m is used as a parcel identifier. The enthalpy difference between the two states is given by

$$\int (h_C - h_A) dm = \int \left(\int_A^C T ds \right) dm + \int \left(\int_A^C \Delta \mu dq \right) dm + \int \left(\int_A^C \alpha dp \right) dm, \quad (10)$$

with \int_A^C indicating an integral following a thermodynamic path between the two reference states. The transformation between A and C can follow an infinite number of such paths. One possibility is to define an intermediary state B which has the same pressure distribution as A , but the same entropy and moisture distribution as C . The transformation between A and C can thus be subdivided in a change of entropy and moisture at constant pressure (leg A to B), followed by a reversible adiabatic pressure adjustment (leg B to C). In this case, we have:

$$\int (h_B - h_A) dm = \int \left(\int_A^B T ds \right) dm + \int \left(\int_A^B \Delta \mu dq \right) dm \quad (11)$$

$$\int (h_C - h_B) dm = \int \left(\int_B^C \alpha dp \right) dm \quad (12)$$

By construction, C is the reference state for B , it minimizes the total static energy, and the integral

$$\int \left(\int_B^C \alpha dp \right) dm \leq 0 \quad (13)$$

is negative. Indeed, if it were positive, B would have a lower total enthalpy than C .

Similarly, the state D is constructed as to have the same pressure distribution as C and the same entropy and total humidity as A . Hence, we have

$$\int (h_D - h_C) dm = \int \left(\int_C^D T ds \right) dm + \int \left(\int_C^D \Delta \mu dq \right) dm \quad (14)$$

$$\int (h_A - h_D) dm = \int \left(\int_D^A \alpha dp \right) dm \quad (15)$$

As A is the reference state for B , we have

$$\int \left(\int_D^A \alpha dp \right) dm \leq 0. \quad (16)$$

Adding (11-12) and (14-15) yields

$$\int \left(\int_B^C \alpha dp + \int_D^A \alpha dp \right) dm = - \int \left(\int_A^B (T ds + \Delta \mu dq) + \int_C^D (T ds + \Delta \mu dq) \right) dm. \quad (17)$$

The constraints (13) and (16) imply that the left-hand side must always be negative.

Instead of the cycle $A-B-C-D$, one can subdivide the path between A and C into $N+1$ intervals by selecting a set of intermediary reference states A_1, A_2, \dots, A_n . As before, construct B_i as the state with the same pressure distribution as A_i and the same entropy and total water distributions as in A_{i+1} . Similarly, the states D_i are constructed with the same pressure distribution as A_{i+1} , and same entropy and total water distributions as A_i . We can now construct the cycle $A=A_0, B_0, A_1, \dots, B_N, C=A_{N+1}, D_{N+1}, A_N, \dots, D_1, A=A_0$. The equivalent of equation (17) includes now the integrals taken over $2N+2$ segments:

$$\int \sum_{i=0}^N \left(\int_{B_i}^{A_{i+1}} \alpha dp \right) dm + \int \sum_{i=0}^N \left(\int_{D_{i+1}}^{A_i} \alpha dp \right) dm = - \int \sum_{i=0}^N \left(\int_{A_i}^{B_i} (Tds + \Delta \mu dq) \right) dm - \int \sum_{i=0}^N \left(\int_{A_{i+1}}^{D_{i+1}} (Tds + \Delta \mu dq) \right) dm \quad (18)$$

By increasing the number of subintervals N , the terms on the right-hand side converge toward the integral following a path of reference state:

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{i=0}^N \left(\int_{A_i}^{B_i} (Tds + \Delta \mu dq) \right) &= - \lim_{N \rightarrow \infty} \sum_{i=0}^N \left(\int_{A_{i+1}}^{D_{i+1}} (Tds + \Delta \mu dq) \right), \\ &= \int_A^C (T_{ref} ds + \Delta \mu_{ref} dq) \end{aligned} \quad (19)$$

where the subscript ref is used to emphasize the fact that an integral is taken along a path of reference states. In the limit of $N \rightarrow \infty$, the right-hand side of (18) vanishes, and we have

$$\lim_{N \rightarrow \infty} \int \sum_{i=0}^N \left(\int_{B_i}^{A_{i+1}} \alpha dp \right) dm = - \lim_{N \rightarrow \infty} \int \sum_{i=0}^N \left(\int_{D_{i+1}}^{A_i} \alpha dp \right) dm \quad (20)$$

As A_0, A_1, \dots, A_{n+1} correspond to reference states, the left-hand side is always negative for finite value of N , and the right-hand side is positive. Therefore, the limit converges toward 0:

$$\lim_{N \rightarrow \infty} \int \sum_{i=0}^N \left(\int_{B_i}^{A_{i+1}} \alpha dp \right) dm = \lim_{N \rightarrow \infty} \int \sum_{i=0}^N \left(\int_{D_{i+1}}^{A_i} \alpha dp \right) dm = \int_A^B \alpha_{ref} dp_{ref} dm = 0 \quad (21)$$

where the integral is taken following a path of reference states.

This result is a direct consequence of the fact that the reference state minimizes the total static energy of the atmosphere for reversible adiabatic transformations. The cancellation of the pressure term in (21) is very similar to the constrained extremum theorem in calculus. Given two smooth functions $F(X)$ and $G(X)$, with X a n -dimensional

vector, so that F has an extremum at X_0 under the constraint that $G(X) = 0$, then the directional derivative of F in any direction allowed by the constraint $G(X)=0$ must vanish:

$$\sum_i \frac{\partial F}{\partial x_i}(X_0) dx_i = 0 \text{ for } \forall dx \in \mathfrak{R}^N \text{ such that } \sum_i \frac{\partial G_k}{\partial x_i}(X_0) dx_i = 0 .$$

In the context of the previous discussion, one can interpret the function F as the total enthalpy of the atmosphere, while the set of constraints G_k corresponds to the requirement that the entropy and total water of each parcel is conserved. For a finite set of parcels, the extremum theorem implies that F the total enthalpy remain unchanged for an infinitesimal adiabatic redistribution of the parcels near a reference state X_0 that minimizes F . The derivation of equation (21) can be viewed as a specialized proof of the extremum principle for an infinite number of parcels and in which the derivatives are expressed in terms of thermodynamic variables.

Equations (21) implies in particular that the reference pressure term in (5) vanishes

$$\int \int \frac{d}{dt} \left(\int_{p_{ref}(t_0)}^{p_{ref}(t_0+t)} \alpha_{ref} dp \right) dm = 0 ,$$

so that the sources and sinks of APE can be written as:

$$\frac{d}{dt} APE = \int (T - T_{ref}) \frac{ds}{dt} dm + \int (\Delta\mu - \Delta\mu_{ref}) \frac{dq}{dt} dm - \frac{d}{dt} KE. \quad (22)$$

The first two terms on the left-hand side are the sources and sinks of APE associated with the phase transition of water and with the external forcing, while the third terms is the conversion of APE into kinetic energy. Before discussing these terms in greater detail in section 3, we need also to address the impact of changing the total water content of a parcel.

2.2. Open transformations

Open transformations here mean transformations in which the total water in a parcel is changed. It is only necessary to consider the addition or removal of a single phase, which in practice can be either preceded or followed by a phase transition. Consider the addition of a quantity dq_l of liquid water to a given parcel. The enthalpy of the parcel changes by

$$dh = h_l dq_l = T ds - \mu_l dq_l \quad (23)$$

where $h_l = C_l (T - T_0)$ is the specific enthalpy of liquid water, $ds = s_l dq_l$ is the entropy change in the parcel, $s_l = C_l \ln \frac{T}{T_0}$ is the specific entropy of liquid water, and T_0 is an arbitrary reference temperature. Note that the change in parcel entropy can be written as the sum of an entropy change and a chemical potential contribution, similar to the formula for a closed transformation.

The effect of the mass change is not limited to the parcel itself. Indeed, for the atmosphere to remain hydrostatic, the pressure below the parcel must increase by $dp = g dq_l \frac{dm}{dA}$, with dA the horizontal area of the parcel. For an adiabatic atmosphere, the compression is associated with an enthalpy increase αdp in all the underlying parcels. Integrated over the entire column, this is equal to the geopotential energy of the water $g Z_p dq_l dm$. Combined with (23), this implies that the total enthalpy change in the atmospheric column is equal to the sum of the enthalpy and geopotential energy of the liquid water.

The change in APE is obtained by taking the difference in the integrated enthalpy between the current and reference states:

$$dAPE = \left[C_l(T - T_{ref} - T_{ref} \ln \frac{T}{T_{ref}}) + g(Z - Z_{ref}) \right] dm dq_l. \quad (24)$$

Note that the contribution of the term $gZ_{ref}dq_l dm$ arises from integrating the changes in the reference pressure:

$$\int \alpha_{ref} dp_{ref} = gZ_{ref} dq_l dm \quad (25)$$

This does not contradict the result from the previous section. Indeed, the reference state is determined by minimizing the total static energy through a succession of reversible adiabatic transformations *while conserving the total water*. As this excludes open transformations, the extremum theorem does not apply to the change of total enthalpy resulting from the addition or removal of water.

It should be stressed here that the expression (24) is valid for the rate of change of APE by the addition of condensed water in a hydrostatic atmosphere. Without this underlying hydrostatic balance, there would be an additional contribution associated with elastic effects (see Bannon 2004).

3. Sources and Sinks of APE

Equations (22) and (24) can be used to compute the impact of various processes on the APE. It should be noted here that while the APE itself is a property of the distribution of entropy and water vapor of the whole atmosphere, the sources and sinks of APE can be computed for specific parcels, independently of the transformations taking place in the rest of the atmosphere.

This section discusses the APE contribution of six main atmospheric processes: external heat source, surface evaporation, diffusion of heat and of water vapor, precipitation, and re-evaporation of condensed water. The APE sources and sinks are also estimated for an idealized radiative-convective equilibrium. This example is meant to be illustrative, and complete calculation of the APE sources and sinks would be performed in a subsequent paper.

3.1. External Heating

An external heat source Q , such as surface sensible heat flux or radiation, is associated with an entropy change of $\frac{Q}{T}$. Hence, the net source of APE is equal to

$$\left. \frac{\partial APE}{\partial t} \right|_{ext} = \int Q \frac{T - T_{ref}}{T} dm. \quad (26)$$

This is equal to the heating rate multiplied by a ‘potential efficiency’. This efficiency is the same as that of a perfect heat engine acting between the parcel’s temperature and its reference state. The potential efficiency is negative when the reference temperature is higher than air parcel. In such case, heating would reduce the APE, while cooling would increase it.

For heating at the surface, the potential efficiency can be extremely large, up to 1/3 for an air parcel at $T \sim 300K$ and with a reference temperature $T_{ref} \sim 200K$. The potential efficiency is however highly sensitive to the parcel’s properties. In particular, heating in a parcel whose reference level remains near the surface would generate little or no APE. Tropospheric cooling can also produce APE if the cooling occurs in parcels whose

reference level is lower than their current locations. Frictional dissipation is associated with a net irreversible entropy production equal to $\frac{D}{T}$, with D the amount of kinetic energy dissipated, so that its effect on APE is the same as a heat source with $Q=D$. As the potential efficiency is always less than 1, frictional heating always produces less APE than the amount of kinetic energy dissipated.

3.2. Surface Evaporation

The evaporation at the Earth's surface can be treated as a net addition of condensed water, followed by an external heating and phase transition.

$$\begin{aligned} \left. \frac{\partial}{\partial t} APE \right|_E &= \int L_v \frac{T - T_{ref}}{T} \frac{dq}{dt} dm + \int R_v T_{ref} (\ln H - \ln H_{ref}) \frac{dq}{dt} dm \\ &+ \int C_l (T - T_{ref} - T_{ref} \ln \frac{T}{T_{ref}}) \frac{dq}{dt} dm + \int g(Z - Z_{ref}) \frac{dq}{dt} dm \end{aligned} \quad (27)$$

The first term on the right-hand side is similar to the contribution of and external in source in (26), and is equal to the net latent heat source multiplied by the potential efficiency. The second term accounts for the variation of chemical potential for water vapor. For surface evaporation, the reference relative humidity will always be higher than the current humidity, and the second term will always be negative. Note that if the parcel is saturated in the reference state, $\ln H_{ref} = 0$, so that the loss of APE is proportional to the irreversible entropy production due to evaporation:

$$R_v T_{ref} (\ln H - \ln H_{ref}) = -T_{ref} \Delta S_{irr} \quad (28)$$

with ΔS_{irr} the irreversible entropy production associated with the evaporation of water in unsaturated air.

The third term on the right-hand side of (27) is associated with the increased heat capacity of the air parcel. A Taylor expansion yields

$$C_l(T - T_{ref} - T_{ref} \ln \frac{T}{T_{ref}}) \approx \frac{1}{2} C_l \frac{(T - T_{ref})^2}{T_{ref}}. \quad (29)$$

This corresponds to the work of a perfect heat engine transporting a quantity of heat $C_l(T - T_{ref})$ over a temperature difference $\frac{1}{2}(T - T_{ref})$. The fourth term corresponds to the work required to lift the additional water from the surface to its reference level.

Consider for example evaporation in an air parcel at the surface, with $T = 300K$, $H = 0.8$, $Z = 0$, and whose reference state is in the upper troposphere, with $T_{ref} = 200K$, $H_{ref} = 1$, $Z_{ref} = 15,000m$. The contribution of the latent heating is $\frac{T - T_{ref}}{T} L_v \approx 750kJ/kg$, the

impact of the chemical potential is $R_v T_{ref} \ln \frac{H}{H_{ref}} \approx -20kJ/kg$, the contribution of the

specific heat of liquid water amounts to $C_l(T - T_{ref} - T_{ref} \ln \frac{T}{T_{ref}}) \approx 80kJ/kg$, and the water

loading is $g(Z - Z_{ref}) \approx 145kJ/kg$. A dimensional analysis indicates that the latent heat term dominates (27), though the combined contribution of the other terms cannot be entirely neglected. Furthermore, as for external heating, the net effect of the evaporation is highly sensitive to a parcel reference state. In particular, evaporation into a “stable” parcel - whose reference state is the same as its current state - would not change the APE.

3.3. Diffusion of heat

Diffusion of heat can be viewed as a combination of a heat source and a heat sink in different parcels. The diffusion of a quantity of heat Q from a parcel A to a parcel B can be obtained through (26):

$$\left. \frac{\partial}{\partial t} APE \right|_{diff, sen} = Q \left(\frac{T_{ref,A}}{T_A} - \frac{T_{ref,B}}{T_B} \right) \approx \frac{T_{ref,A} - T_{ref,B}}{T} Q \quad (30)$$

where it has been assumed that the parcel temperatures are similar $T_A \approx T_B \approx T$.

In a dry atmosphere, the reference temperature of an air parcel is determined by its potential temperature. If diffusion takes place between parcel's at the same pressure, the term $\frac{T_{ref,A} - T_{ref,B}}{T}$ is proportional to the buoyancy difference between the air parcels.

This implies that, outside very specific circumstances, diffusion of heat will be limited to exchange between air parcels with similar reference temperatures.

In a moist atmosphere, the reference temperature of parcels at same temperature (or virtual temperature) and pressure can vary significantly because of water vapor. In particular, it is possible for a moist air parcel, with a reference level in the upper troposphere, can have the same buoyancy as a comparatively dry parcel with a reference level near the Earth's surface. This greatly enhances the impact of diffusion on APE in a moist atmosphere.

3.4. Diffusion of water vapor

The APE changes due to diffusion of water vapor can be obtained by applying equation (27) to a balanced water vapor source and sink. The main contribution is associated with the latent heat transport

$$\left. \frac{d}{dt} APE \right|_{diff, v} \approx \left(\frac{T_{ref,A}}{T_A} - \frac{T_{ref,B}}{T_B} \right) L_v dqdm \approx \frac{T_{ref,A} - T_{ref,B}}{T} L_v dqdm. \quad (31)$$

This main contribution is equal to the APE changes associated with a sensible heat flux of the same amplitude as the latent heat transport. As the diffusive flux of latent heat is usually an order of magnitude larger than sensible heat, the diffusion of water vapor has

the potential to be the largest sink of APE in the atmosphere. For example, the diffusion of 1 g of water vapor from an “unstable” air parcel with $T_{ref,A} \approx 200K$ to a “stable” parcel with $T_{ref,B} \approx 200K$ would result in a destruction of APE of approximately 500 Joules (accounting for all the terms, not only the latent heat contribution).

3.5. Precipitation

Falling precipitation can be treated by transferring condensed water from one air parcel to another one, until it reaches the ground. Expression (24) indicates that the removal of condensed water will affect the APE both through its effect on the heat capacity (first term on the right-hand side of 24) and geopotential (the second term). For precipitation falling out of a convective updraft (assuming thus $T > T_{ref}, Z < Z_{ref}$), the reduction in heat capacity reduces the APE, while the reduction in water loading increases the APE. The positive contribution of the water loading term can be seen as the result of the enhanced buoyancy due to a reduction of water loading. Its effect is equal to the difference in geopotential energy between the removed water’s current and reference states. A comparison between the terms in (24) indicates that the water loading is the dominant effect, so that the removal of precipitating water corresponds to a net source of APE for the atmosphere. This is a remarkable aspect of moist atmospheres: an adiabatic process can increase the APE without affecting the kinetic energy of the atmosphere.

The role of the water loading on APE can be better understood by looking at the impact of adding water to a parcel at the Earth’s surface, lifting the parcel to higher level, and removing the water. If the ascent is reversible, its reference level is unchanged, and the net effect of water loading is a reduction of APE, with $dAPE = g(Z_{in} - Z_{out})dqdm$.

This is the amount of work required to lift the water to the level at which it is removed. If the precipitation falls through air at rest, this is also the amount of kinetic energy dissipated in the microscopic shear zones surrounding the falling precipitation, as discussed in Pauluis et al. (2000).

3.6. Re-evaporation

The change in APE associated with a re-evaporation of condensed water is given by

$$dAPE = R_v T_{ref} \left(\ln \frac{H}{H_{ref}} \right) dq \quad (32)$$

For an air parcel with a reference level lower than the current level, so $H > H_{ref}$, re-evaporation would increase the APE. Re-evaporation of precipitation in a downdraft increases the APE. In contrast, if the reference level is higher than the current level, for example in the case of re-evaporation in the subcloud layer, then re-evaporation decreases the APE.

It is useful to compare the re-evaporation and diffusion of water vapor. Consider a rising updraft in an unsaturated environment. Re-evaporation can result as some condensed water drifts from the updraft into the environment. A similar effect can be achieved if water vapor diffuses directly from the updraft to the environment. From a thermodynamic point of view, both the re-evaporation and diffusion are irreversible, and are associated with exactly the same entropy production. Yet, from the point of view of APE, diffusion of water vapor would be associated with a large destruction of APE (assuming that the reference temperature of the updraft is lower than the environment in Eq. 32), while the re-evaporation could potentially act as a source. The difference arises

from the fact that the diffusion reduces the upward heat transport by the updraft, while the re-evaporation can increase the net upward heat transport by generating a cold downdraft. This example indicates that the production or destruction of APE is not directly related to the dissipative nature of the process.

3.7. Idealized Radiative-Convective equilibrium

To provide a quantitative example, the sources and sinks of APE for idealized radiative-convective equilibrium are considered. This example considers a quasi-steady atmosphere, heated at the surface with a surface latent heat flux of 90 Wm^{-2} , and a sensible heat flux of 10 Wm^{-2} . This is balanced by a net tropospheric cooling of 100 Wm^{-2} . The surface temperature is 300 K , and the tropopause temperature is 200 K . These numbers are chosen to be illustrative of tropical conditions. The reference state can be constructed by lifting the “unstable” parcels from the boundary layer to the tropopause, while the rest of the atmosphere, - the “stable” parcels – is only compressed during the ascent of the unstable parcels. The portion of the atmosphere that rises up to the upper troposphere cannot be determined a priori, but it is reasonable to assume here that it corresponds to the mass of the subcloud layer, approximately 50 mb .

For the surface flux, one can assume that the surface energy fluxes are transferred across a thin surface layer to “unstable” air parcels. In this case, the effective efficiency for the sensible heat flux is $1/3$, and the source of APE is $\left. \frac{d}{dt} APE \right|_{sen} = 3.3 \text{ Wm}^{-2}$. Using the same value as the example in Section 3.2 for the latent heat flux, the source of APE associated with the latent heat flux to be $\left. \frac{d}{dt} APE \right|_E = 23.8 \text{ Wm}^{-2}$. This corresponds to a

lower efficiency (APE production per unit of energy) for the latent heat flux than for the sensible heat flux. This is primarily due to the water loading in (27). The effect of the radiative cooling depends on its vertical structure and on the portion of the atmosphere that is unstable. If the mass of unstable parcels corresponds to an atmospheric layer of 50mb, the reference temperature of a stable parcel is higher by approximately 5K than its current temperature. For a uniformly distributed radiative cooling, the energy loss in the stable parcels is 95 Wm^{-2} , which, translates into a source of APE of approximately 2 Wm^{-2} . The rest of the radiative cooling, 5 Wm^{-2} acts on unstable parcel, with $T \sim 300 \text{ K}$ and $T_{ref} \sim 200 \text{ K}$, and removes about 1.7 Wm^{-2} of APE. The net contribution of the radiative cooling is small, less than $\left. \frac{d}{dt} APE \right|_{rad} < 0.5 \text{ Wm}^{-2}$. There is a near cancellation between the cooling of the stable and unstable parcels, which results from the choice of a uniform cooling rate and the fact that the mean temperature of the reference state is usually close to the mean atmospheric temperature.

The contribution of internal processes requires further assumption, and should thus be accepted as tentative until further study. While precipitation is a dissipative mechanism, it is shown in section 3.5 that it acts as a source of APE, when precipitation forms in parcels that are lower than their reference level. In our example, assuming that the precipitation falls on average from 5000m yields a source of APE given by

$$\left. \frac{d}{dt} APE \right|_{prec} \sim P g (Z - Z_{ref}) \sim 3.6 \text{ Wm}^{-2}, \text{ with } P \text{ the precipitation rate. Re-evaporation can act as}$$

well as a source or as a sink of APE depending on where it occurs. A dimensional scaling of (32) yields a value of $\left. \frac{d}{dt} APE \right|_{re-ev} \sim P R_v T \sim 3 \text{ Wm}^{-2}$. The sign of the contribution is

unknown, and this value most likely overestimates the impact of re-evaporation. The impact of diffusion can be obtained if one considers that a fraction ε of the surface fluxes is diffused from unstable parcels to stable parcels. In this case, the APE destroyed by diffusion of heat and water vapor would be given by

$$\left. \frac{d}{dt} APE \right|_{diff} \approx -\varepsilon \left. \frac{d}{dt} APE \right|_{sfc},$$

with $\left. \frac{d}{dt} APE \right|_{sfc} = \left. \frac{d}{dt} APE \right|_{sen} + \left. \frac{d}{dt} APE \right|_E$ the contribution of the surface fluxes.

The APE production due to the external heat sources and sinks is close to $\left. \frac{d}{dt} APE \right|_{sfc} \approx 28 Wm^{-2}$. This is extremely large both in comparison to the observed atmospheric dissipation of $2 - 4 Wm^{-2}$ (see for example Peixoto and Oort, 1993) or results from numerical simulations of radiative-convective equilibrium such as Pauluis and Held (2002a) where the kinetic energy production and dissipation in convective motions is about $1 Wm^{-2}$. This difference cannot be explained by the contribution of re-evaporation or precipitation (the latter is a source of APE in any case). The only significant sink of APE is diffusion of heat and water vapor. The net APE source due to the surface heat flux and diffusion is approximately

$$\left. \frac{d}{dt} APE \right|_{sfc} + \left. \frac{d}{dt} APE \right|_{diff} \approx Q_{surf} \frac{T_{surf} - T_{trop}}{T_{surf}} (1 - \varepsilon), \quad (33)$$

where Q_{surf} is the total surface energy flux, and T_{surf} and T_{trop} are the surface and tropopause temperatures respectively. (The contribution of evaporation has been approximated as that of an external heat source.) The difference between the large APE source due to surface flux, and the weak observed dissipation can only be reconciled if,

after diffusion is taken into account, only a small fraction of the surface fluxes is used to increase the energy of stable parcels, which implies $\varepsilon \sim 0.8 \dots 1$.

4. Discussion

In this paper, an analytic formula for the sources and sinks of APE has been derived that is equally valid for dry and moist atmospheres. A key element of this derivation lies in proving that the contribution due to the reorganization of air parcels in the reference state vanishes. This is a direct consequence of choosing a reference state that minimizes the total static energy of the atmosphere, and is therefore valid only in the context of Lorenz APE framework. It is worth stressing that despite the widespread use of APE to discuss the maintenance of the atmospheric circulation, this is the first time that an analytic expression for the production and dissipation of APE in a moist atmosphere has been explicitly derived.

The expression for the APE production is such that the contribution of individual processes can be identified, independently of the transformations taking place in other parts of the atmosphere. This has been used here to discuss the contribution of external heat sources, diffusion of water vapor, surface evaporation, precipitation and diffusion of water vapor.

The production of APE by an external heat source is given by equation (26), and is equal to the net heat source multiplied by a potential efficiency. The potential efficiency is equal to the efficiency of a Carnot cycle with an energy source at the current parcel temperature and an energy sink at the parcel temperature in its reference state. For an energy source at the surface, the same energy flux will generate more APE if it occurs in a parcel with low reference temperature than in an air parcel with high reference

temperature. This indicates that the generation of APE is maximized for heating in ‘unstable’ air parcels, i.e. parcels whose reference level is situated in the upper troposphere, while heating ‘stable’ parcels, i.e. whose reference level is situated near the surface, has little impact on the APE.

A comparison between ‘dry’ and ‘moist’ processes indicates that the differences are related primarily to the effects of the water loading, the heat capacity of liquid water, and to the chemical potential difference between water vapor and liquid water. These contributions are in general small when an energy source or transport is involved. The dominant contribution of a latent heat flux is the same as that of a corresponding sensible heat flux. The primary difference between dry and moist atmosphere is not in the expression for the APE sources and sinks, but lies in the fact that, in a moist atmosphere, parcels with same specific volume and pressure can have a very different static energy and thus reference temperature. For this reason, one expects diffusion to play a major role in the APE budget.

Estimates of the APE production for an idealized example have been considered. For typical conditions, the surface energy flux would produce a very large source of APE if all the heating occurs in unstable air parcels, close to 30Wm^{-2} for a surface flux of 100Wm^{-2} . This APE source is much larger than typical estimate of the work produced and dissipated on average by atmospheric motion. It is argued here that this implies that a large fraction of the surface energy flux must end up after diffusion in stable parcels. If only a small fraction $1-\varepsilon$ of the surface flux heats up unstable parcels, the generation of APE can be approximated by

$$\frac{d}{dt} APE \approx Q_{surf} \frac{T_{surf} - T_{trop}}{T_{surf}} (1 - \varepsilon). \quad (34)$$

The net generation of APE due to the surface energy flux and diffusion is strongly controlled by how much mixing takes place between stable and unstable air in the boundary layer. The same surface flux would produce very little APE in subsidence regions, where there is little to no unstable air and ϵ is close to 1, but can produce tremendous amount of APE in convectively active regions such as a hurricane eyewall.

References:

- Bannon, P.R., 2004: Lagrangian Available Energetics and Parcel Instabilities. *J. Atmos. Sci.*, **61**, 1754–1767.
- Bannon, P.R., 2005: Eulerian Available Energetics in Moist Atmosphere. Submitted to *J. Atmos. Sci.*
- Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. *Tellus*, **7**, 157-167.
- Lorenz, E. N., 1978: Available energy and the maintenance of a moist circulation. *Tellus*, **30**, 15-31.
- Pauluis, O. and I.M. Held, 2002a: Entropy budget of an atmosphere in radiative-convective equilibrium. Part I: Maximum work and frictional dissipation. *J. Atmos. Sci.*, **59**, 140-149.
- Pauluis, O. and I.M. Held, 2002b: Entropy budget of an atmosphere in radiative-convective equilibrium. Part II: Latent heat transport and moist processes. *J. Atmos. Sci.*, **59**, 125-139.
- Pauluis, O., V. Balaji and I.M. Held, 2000: Frictional dissipation in a precipitating atmosphere. *J. Atmos. Sci.*, **57**, 989-994.
- Peixoto, J.P. and A.H. Oort, 1993: *Physics of Climate*. American Institute of Physics, 564 pp.
- Randall, D. A., and J. Wang, 1992: The moist available energy of a conditionally unstable atmosphere. *J. Atmos. Sci.*, **49**, 240-255.

List of Table:

Table 1: Sources and Sinks of Available Potential Energy in an idealized radiative-convective equilibrium. See text for details.

| Process | APE contribution |
|----------------------------|---|
| Surface sensible heat flux | $\left. \frac{d}{dt} APE \right _{sen} \approx 3.2 Wm^{-2}$ |
| Surface evaporation | $\left. \frac{d}{dt} APE \right _E \approx 23.8 Wm^{-2}$ |
| Radiative cooling | $\left. \frac{d}{dt} APE \right _{rad} \approx 0.5 Wm^{-2}$ |
| Precipitation | $\left. \frac{d}{dt} APE \right _{prec} \approx 3.6 Wm^{-2}$ |
| Re-evaporation | $-3 Wm^{-2} < \left. \frac{d}{dt} APE \right _{re-ev} < 3 Wm^{-2}$ |
| Diffusion | $\left. \frac{d}{dt} APE \right _{diff} \approx -\epsilon \left(\left. \frac{d}{dt} APE \right _{sen} + \left. \frac{d}{dt} APE \right _E \right)$ |

Table 1: Sources and Sinks of Available Potential Energy in an idealized radiative-convective equilibrium. See text for details.

