Fluid Physics
8.292J/12.330J

Problem Set 1

Solutions

1. *Consider a planetary atmosphere consisting of an ideal gas of gas constant R in which the temperature decreases linearly upward from the surface:*

$$T = T_s - \Gamma z\,,$$

where T_s *is the surface temperature and* Γ *is the gradient of temperature with altitude (sometimes called the lapse rate). Assume that the atmosphere is at rest within a gravitational field characterized by a uniform gravitational acceleration g, and that the surface pressure has the value* p_0. *Derive an expression for the atmospheric pressure as a function of altitude. At what altitude does the pressure vanish? Why?*

Solution: The hydrostatic equation for an ideal gas at rest in a uniform gravitational field can be written

$$\frac{dp}{dz} = -\rho g = -\frac{pg}{RT} = -\frac{pg}{R\left(T_s - \Gamma z\right)}\,. \tag{1}$$

This can be re-arranged to yield

$$\frac{dp}{p} = -\frac{gdz}{R\left(T_s - \Gamma z\right)}. \tag{2}$$

Integrate this up from the surface ($z = 0$, $p = p_0$) to yield

$$p(z) = p_0\left(1 - \frac{\Gamma z}{T_s}\right)^{g/R\Gamma}. \tag{3}$$

The quantity $g/R\Gamma$ is about 5.5 in the Earth's troposphere, which extends up to about 10 km. Given a surface temperature of 280 K, this implies that at 10 km altitude, the pressure is about 25% of its surface value.

If the temperature continued to decrease linearly with height, it would reach absolute zero at an altitude of T_s/Γ. Provided that $g/R\Gamma > 1$, the ideal gas law implies that the density would vanish at this altitude, so the conditions for a well-posed statistical description of the gas would break down before this altitude is reached. In the case of the earth, the temperature stops falling with height at the tropopause (around 10 km altitude) and slowly increase in the stratosphere, reaching a maximum in the upper stratosphere.

2. *A cube of incompressible fluid of density* ρ_c *with massless walls is suspended within another incompressible fluid of density* ρ_a *by means of a massless string attached to wall. The entire system is at rest within a gravitational field of uniform gravitational acceleration g. The length of any side of the cube is* Δx*. Derive an expression for the net force acting on the massless string. Now the string is cut. Find the acceleration of the cube at the moment the string is cut.*

Solution: The pressure force acting on the bottom side of the cube is just the pressure at that altitude times the area of that face of the cube:

$$F_b = p(z)\left(\Delta x\right)^2. \tag{4}$$

Likewise, the force on the upper face of the cube is

$$F_t = -p(z+\Delta x)\left(\Delta x\right)^2. \tag{5}$$

But since the pressure in the environment of the cube is in hydrostatic equilibrium,

$$p(z) = p(z+\Delta x) + \rho_a g \Delta x. \tag{6}$$

Combining (4)-(6) gives

$$F_p = \rho_a g\left(\Delta x\right)^3, \tag{7}$$

where F_p is the net pressure force on the cube.

The weight of the cube is just its density times its volume times gravity:

$$F_w = -\rho_c g\left(\Delta x\right)^3. \tag{8}$$

Adding (7) to (8) gives the net force on the cube:

$$F = \left(\rho_a - \rho_c\right) g\left(\Delta x\right)^3. \tag{9}$$

This is a statement of Archimedes proposition that the buoyancy of a submerged object is the difference between its weight and that of the fluid it displaces.

3. (***Extra credit***) *A completely rigid cylinder in a uniform gravitational field is completely filled with an Euler fluid of density ρ which is at rest. Within the fluid is a small balloon filled with an ideal gas. This balloon is attached to a string whose volume is negligible, by which means the balloon's vertical position can be changed. The tension of the balloon's material can be neglected, so the gas may be taken to be in pressure equilibrium with the adjacent Euler fluid. At the initial time, t_0, the fluid pressure at the position of the balloon is p_0 at an altitude of z_0 above the bottom of the cylinder.*

The balloon is then slowly raised or lowered to a new position z_1 at time t_1. Describe the pressure distribution in the fluid at time t_0 and at time t_1.

Solution: Because the cylinder is completely rigid, its volume cannot change; nor can that of the fluid since it is completely incompressible. Thus the volume of the balloon must be constant, since the sum of it and the volume of liquid must equal that of the cylinder. If the balloon is in thermodynamic equilibrium with its environment, then it will have the temperature of its environment, and the ideal gas law then implies that the pressure of the balloon is invariant.

In solving the hydrostatic equation for the pressure distribution in the liquid, there is an integration constant which we are used to thinking of as the pressure at some free surface. In this case, there is no free surface, and the integration constant is determined by the requirement that the pressure at the height of the balloon is invariant. Thus the pressure distribution is given by

$$p(z) = p_0 + \rho g \left(z_b - z \right),$$

where z_b is the altitude of the balloon at any given time. Thus the pressure distribution at time t_0 is just $p(z) = p_0 + \rho g \left(z_0 - z \right)$, while at time t_1 it is $p(z) = p_0 + \rho g \left(z_1 - z \right)$.