# Fluid Physics
**8.292J/12.330J**

Problem Set 5

**1.** Consider the flow of an Euler fluid in the *x* direction given by

$$\overline{U} = \begin{cases} 0 & \text{for} \quad y > d \\ U_0\left(1 - \dfrac{y}{d}\right) & \text{for} \quad 0 \leq y \leq d \\ U_0\left(1 + \dfrac{y}{d}\right) & \text{for} \quad y < 0 \end{cases}$$

This flow does not vary in *x* or in *z*. Determine the propagation and/or growth/decay of small perturbations to this flow that are sinusoidal in the *x* direction and which do not vary in the *z* direction.

2. Consider a cylindrical tank of radius $r_0$ of an incompressible fluid, rotating at angular velocity $\Omega$ and subject to a uniform gravitational acceleration, $g$, downward along the rotation axis, as pictured below:
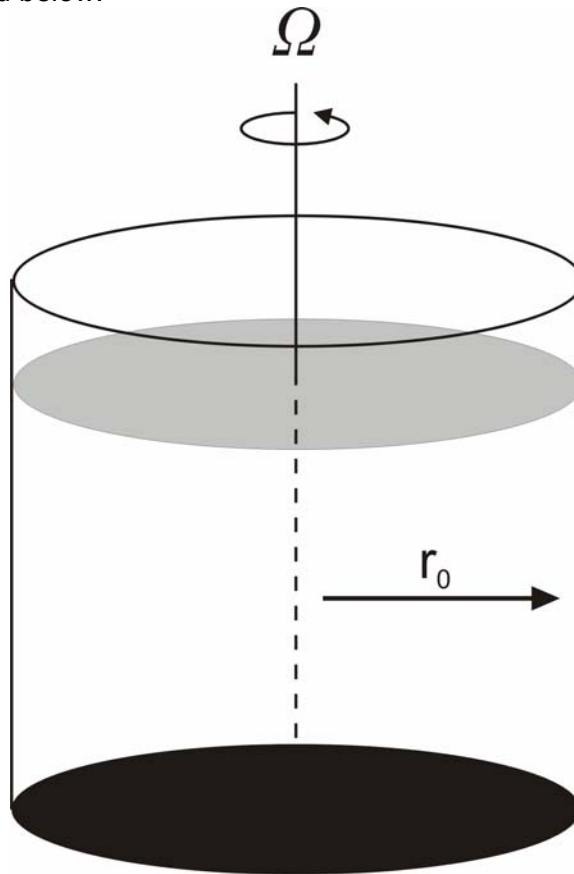
*a.)* Derive an expression for the shape of the free surface.

*b.)* Suppose that the bottom of the cylinder is stationary, so that the fluid is moving with respect to it. This relative motion exerts a torque on the fluid, but we can consider that this torque is confined to a thin layer adjacent to the bottom of the tank. We further assume that the flow remains circularly symmetric. Taking $M$ to be the angular momentum of the fluid per unit mass, $M = rV$ where $V$ is the tangential component of velocity. An equation for the conservation of angular momentum in the thin boundary layer is

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial r} = -r\frac{\partial \tau}{\partial z}, \tag{1}$$

where $\tau$ is called the "frictional stress". We assume that this stress vanishes at the top of the boundary layer (in fact, that is the definition of the top of the boundary layer), and that the stress at the surface is given by

$$\tau_s = -C_D V^2. \tag{2}$$

It can be shown that if the boundary layer is sufficiently thin, the right side of (1) will be large compared to the first term on the left, and so, to a good approximation,

$$u\frac{\partial M}{\partial r} \cong -r\frac{\partial \tau}{\partial z} \tag{3}$$

We can further assume that $M$ does not vary with altitude within the thin boundary layer.

The incompressible mass continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \tag{4}$$

Taking $w$ to vanish at the bottom of the tank, use (2) - (4) to find an expression for $w$ at the top of the boundary layer, in terms of the radial distribution of $V$ (or $M$). Evaluate this expression for the case that $V = V_0\, {r}/{r_0}$.

*c.)* One problem with this solution is that it requires radial flow through the outer wall of the tank. Consider instead an unbounded tank (let $r_0 \to \infty$), but use the following velocity distribution:

$$V = \begin{cases} V_m \left( r / r_m \right), & r \leq r_m \\ \\ V_m \left( r_m / r \right)^n, & r > r_m \end{cases} \tag{5}$$

where $n$ is a number between zero and one. Find the depth-averaged radial velocity in the boundary layer and the vertical velocity at the top of the boundary layer.

*d.)* Extra credit: Considering the depth of the tank to be $H$, which may be taken to be large compared to the thickness of the boundary layer, find an expression for the instantaneous rate of decrease (with time) of angular momentum of the fluid above the boundary layer. What value of $n$ makes this rate of decrease independent of radius?