Fluid Physics
8.292J/12.330J

Problem Set 5

Solutions

1. *Consider the flow of an Euler fluid in the x direction given by*

$$\overline{U} = \begin{cases} 0 & \textit{for} \quad y > d \\ U_0\left(1 - {y}/{d}\right) & \textit{for} \quad 0 \le y \le d \\ U_0\left(1 + {y}/{d}\right) & \textit{for} \quad y < 0 \end{cases}$$

This flow does not vary in x or in z. Determine the propagation and/or growth/decay of small perturbations to this flow that are sinusoidal in the x direction and which do not vary in the z direction.

Solution:

We begin with the general equation for the structure in y of two-dimensional disturbances:

$$\frac{d^2\tilde{v}}{dy^2} - \tilde{v}\left[k^2 + \frac{\overline{U}_{yy}}{\overline{U} - c}\right] = 0. \tag{1}$$

Note that, as in the classical Rayleigh problem, the second term in brackets vanishes within each of the three regions of the flow. As in that problem, we find solutions to (1) within each region and match across the two boundaries separating the regions.

The solution to (1) in each region, which satisfy the boundary conditions at $y = \pm\infty$ have the form

$$\begin{aligned} \tilde{v} &= Ae^{-ky}, & y &> d, \\ \tilde{v} &= Be^{-ky} + Ce^{ky}, & 0 &\le y \le d, \\ \tilde{v} &= De^{ky}, & y &< 0. \end{aligned} \tag{2}$$

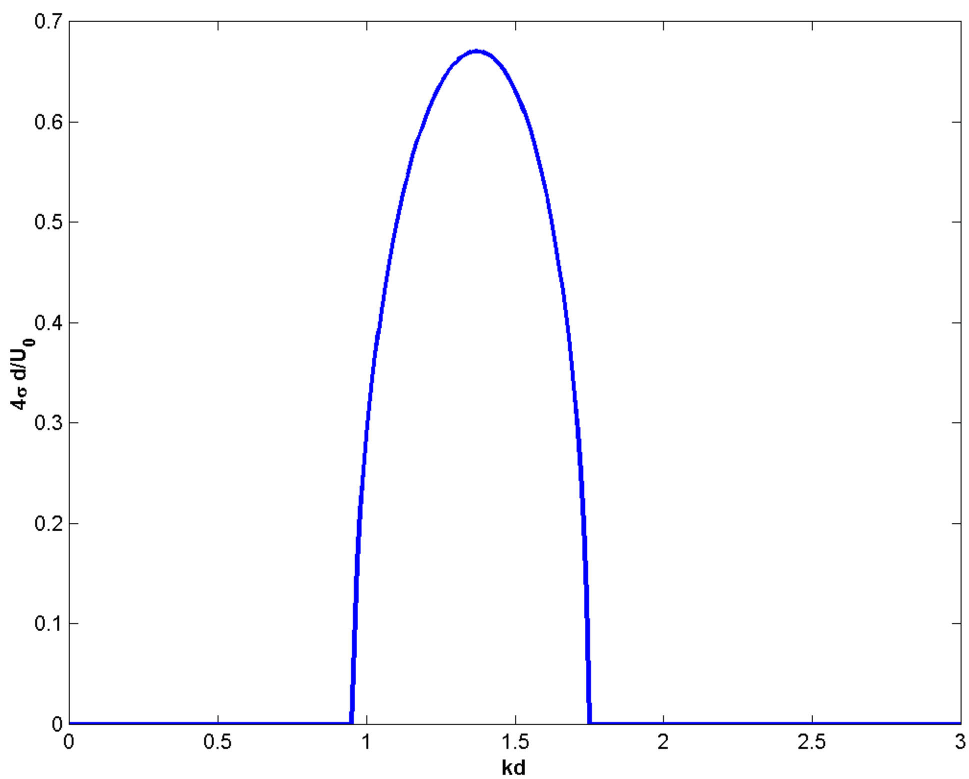
The first matching condition is to match the displacement in y, which in this problem is equivalent to matching $\tilde{v}$ itself. The second condition is to match the fluid pressure, which can be found from the original linearized equations or from integrating (1) in y to get continuity of

$$\frac{d\tilde{v}}{dy} - \tilde{v}\frac{\bar{U}_y}{\bar{U} - c}.$$

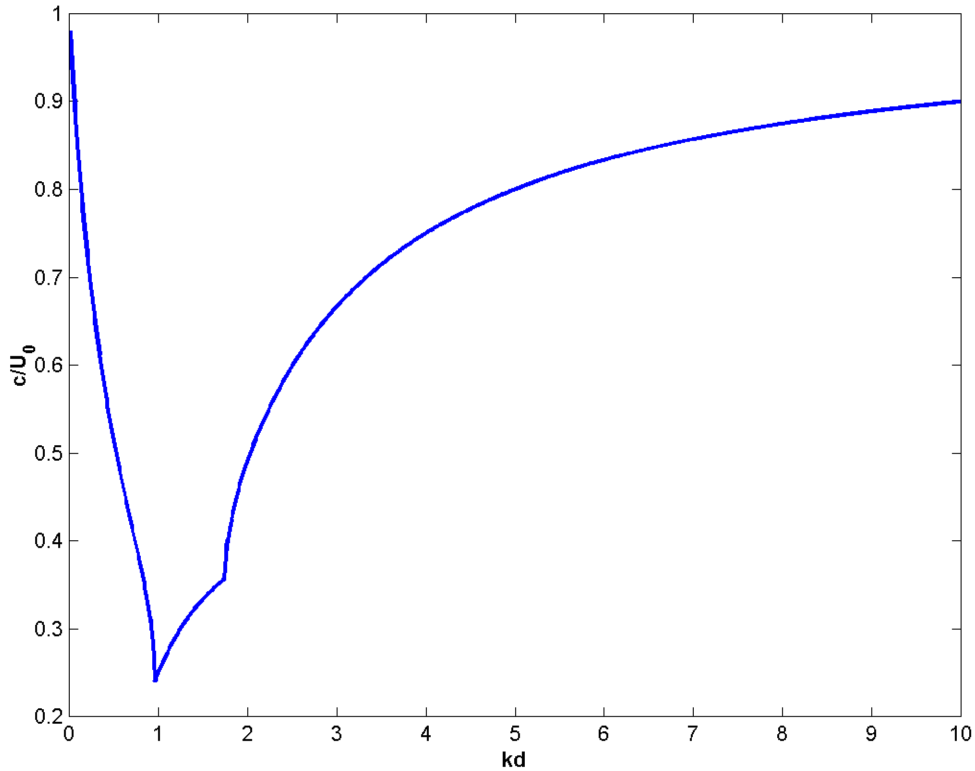
Applying these two matching conditions at $y = 0$ and $y = d$ to (2) yields the dispersion relation

$$\frac{4ckd}{U_0} = 2kd - 1 \pm \sqrt{9 - 12kd + 4(kd)^2 - 8e^{-2kd}} \quad (3)$$

There is a range of kd in which the argument of the square root in (3) is negative; in this range, c had an imaginary part and the solutions are exponentially growing or decaying. The figure below shows the positive root of the nondimensional growth rate, σ:



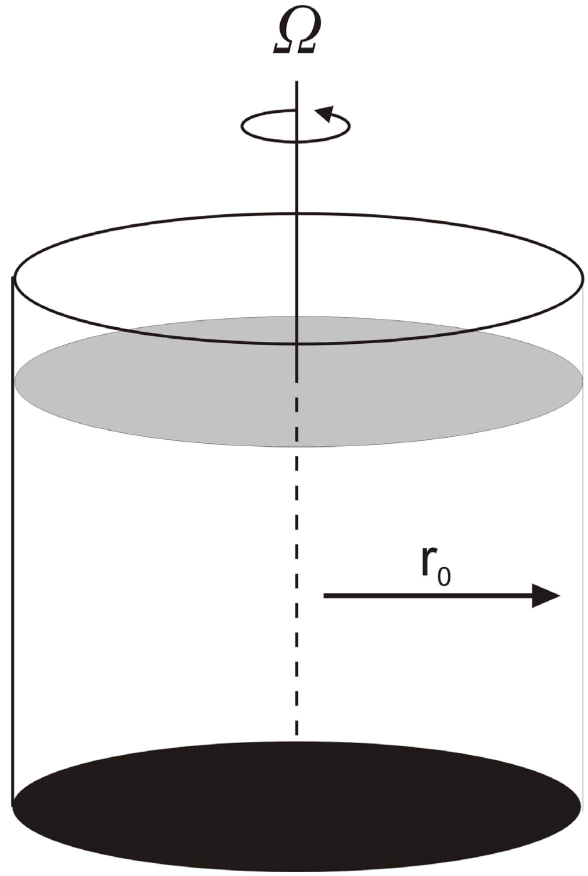
The upper root of the solution for the nondimensional phase speed over a larger range of kd is shown below:



Note that the phase speeds are always smaller than the peak velocity of the mean flow.

Like the classical Rayleigh problem, the two transition regions of the mean flow support Rossby waves which, in a narrow regime of wavelengths, may interact unstably to produce exponentially amplifying disturbances.

***2.** Consider a cylindrical tank of radius r_0 of an incompressible fluid, rotating at angular velocity Ω and subject to a uniform gravitational acceleration, g, downward along the rotation axis, as pictured below:*



a.) Derive an expression for the shape of the free surface.

Solution: In equilibrium, the radial pressure gradient acceleration balances the outward centrifugal acceleration:

$$g\frac{dh}{dr} = \frac{V^2}{r} = \Omega^2 r,$$

which, when integrated, yields

$$h = h_0 + \frac{\Omega^2}{2g} r^2,$$

where h_0 is the height of the surface at the center. Thus the surface is parabolic.

b.) Suppose that the bottom of the cylinder is stationary, so that the fluid is moving with respect to it. This relative motion exerts a torque on the fluid, but we can consider that this torque is confined to a thin layer adjacent to the bottom of the tank. We further assume that the flow remains circularly symmetric. Taking M to be the angular momentum of the fluid per unit mass, $M = rV$ where V is the tangential component of velocity. An equation for the conservation of angular momentum in the thin boundary layer is

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial r} = -r\frac{\partial \tau}{\partial z}, \tag{4}$$

where τ is called the "frictional stress". We assume that this stress vanishes at the top of the boundary layer (in fact, that is the definition of the top of the boundary layer), and that the stress at the surface is given by

$$\tau_s = -C_D V^2. \tag{5}$$

It can be shown that if the boundary layer is sufficiently thin, the right side of (4) will be large compared to the first term on the left, and so, to a good approximation,

$$u\frac{\partial M}{\partial r} \cong -r\frac{\partial \tau}{\partial z} \tag{6}$$

We can further assume that M does not vary with altitude within the thin boundary layer.

The incompressible mass continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \tag{7}$$

Taking w to vanish at the bottom of the tank, use (5) - (7) to find an expression for w at the top of the boundary layer, in terms of the radial distribution of V (or M). Evaluate this expression for the case that $V = V_0 \, {}^{r}\!/_{r_0}$.

Solution: If we integrate *(6)* through the depth of the boundary layer and apply *(5)*, we get

$$uh = -\frac{rC_D V^2}{\partial M/\partial r}, \tag{8}$$

where *h* is the boundary layer depth. Now integrating *(7)* over the depth of the boundary layer gives

$$w_b = -\frac{1}{r}\frac{\partial}{\partial r}(ruh), \tag{9}$$

where w_b is the vertical velocity at the top of the boundary layer. Combining (8) and (9) gives

$$w_b = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r^2 C_D V^2}{\partial M / \partial r}\right]. \tag{10}$$

In the case that $V = V_0 \, r/r_0$, this gives

$$w_b = \frac{3}{2} C_D V_0 \, r/r_0 \,. \tag{11}$$

Here it is seen that the vertical velocity is, in this case, proportional by a constant to the swirling velocity component.

c.) One problem with this solution is that it requires radial flow through the outer wall of the tank. Consider instead and unbounded tank (let $r_0 \to \infty$), but use the following velocity distribution:

$$V = \begin{cases} V_m \left(r/r_m \right), & r \le r_m \\ V_m \left(r_m/r \right)^n, & r > r_m \end{cases} \tag{12}$$

where n is a number between zero and one. Find the depth-averaged radial velocity in the boundary layer and the vertical velocity at the top of the boundary layer.

Solution: For the interior part of this flow, the solution is, of course, given by (8) and (11). Substituting the second part of (12) into (8) and (10) gives

$$uh = -\frac{C_D}{1-n} V_m r \left(\frac{r_m}{r}\right)^n, \tag{13}$$

and

$$w_b = C_D V_m \frac{2-n}{1-n}\left(\frac{r_m}{r}\right)^n. \tag{14}$$

d.) Extra credit: Considering the depth of the tank to be H, which may be taken to be large compared to the thickness of the boundary layer, find an expression for the instantaneous rate of decrease (with time) of angular momentum of the fluid above the boundary layer. What value of n makes this rate of decrease independent of radius?

Solution: According to the Taylor-Proudman Theorem, in a slowly changing, rotating, incompressible and inviscid fluid, the velocity should not vary along the direction of the rotation vector. Accordingly, above the boundary layer in the present flow, the radial and swirling velocity components should not vary with height. If we use U to denote the radial velocity above the boundary layer, mass conservation tells us that there can be no net mass flow across any cylindrical surface, so that

$$UH = -uh, \tag{15}$$

where H is the fluid depth (technically, minus the boundary layer depth, *h*). Applying this to the outer solution, (13), gives

$$U = \frac{C_D}{H(1-n)} V_m r \left(\frac{r_m}{r}\right)^n . \tag{16}$$

Now angular momentum is conserved in the inviscid flow above the boundary layer, so the time rate of change of angular momentum is given by

$$\frac{\partial M}{\partial t} = -U\frac{\partial M}{\partial r}. \tag{17}$$

Substituting (16) for U and using (12) to calculate $\partial M / \partial r$ in (17) gives

$$\frac{\partial M}{\partial t} = -\frac{C_D}{H} V_m{}^2 r \left(\frac{r_m}{r}\right)^{2n} . \tag{18}$$

Clearly, when $n = 1/2$, this spin down rate is independent of radius.