Basics of pressure coordinates

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From time to time, I try to distribute some handouts which I think would help clarify the material. Rob Korty, the former TA for this course, also created useful notes, and I will hand them out with his permission. This is the first of such attempts, intended mainly for oceanographers.

Ways to remember hydrostatic balance in *p* coordinates

As p depends only on z if t, x, and y are fixed, we can write

$$\left(\frac{\partial p}{\partial z}\right)_{t,x,y} = -\rho g\,, \qquad \left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho g}\,.$$

This can be easily transformed to the equation we saw in the class:

$$g\left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho}\,, \qquad \frac{\partial}{\partial z}\left(\int g dz\right)\left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho}\,,$$

$$\frac{\partial \varphi}{\partial z}\left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho}\,, \qquad \left(\frac{\partial \varphi}{\partial p}\right)_{t,x,y} = -\alpha\,.$$

For me, the one easiest to remember is

$$\left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho g} \qquad \text{or} \qquad g\left(\frac{\partial z}{\partial p}\right)_{t,x,y} = -\frac{1}{\rho}\,.$$

So find your favorite.

On $\alpha \nabla_z p = \nabla_p \varphi$

A change in p can be written as

$$dp = \frac{\partial p}{\partial t} dt + \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz .$$

Usually this equation is used to calculate dp from $dt, dx, dy,$ and dz. However, we may use it to constrain $dt, dx, dy,$ and dz by specifying dp. Now, suppose $dt = dy = 0$ and $dp = 0$. Then,

$$dp = 0 = \left(\frac{\partial p}{\partial x}\right)_{t,y,z} dx + \left(\frac{\partial p}{\partial z}\right)_{t,x,y} dz , \qquad \left(\frac{\partial p}{\partial z}\right)_{t,x,y} dz = -\left(\frac{\partial p}{\partial x}\right)_{t,y,z} dx ,$$

$$\frac{dz}{dx} = -\frac{\left(\frac{\partial p}{\partial x}\right)_{t,y,z}}{\left(\frac{\partial p}{\partial z}\right)_{t,x,y}} .$$

Since the left-hand side of the above equation denotes dz/dx with $dt, dy, dp = 0$, we arrive at

$$\left(\frac{\partial z}{\partial x}\right)_{t,y,p} = -\frac{\left(\frac{\partial p}{\partial x}\right)_{t,y,z}}{\left(\frac{\partial p}{\partial z}\right)_{t,x,y}} . \tag{*}$$

Some of you have already noticed that the algebra so far is merely a casual way to derive the implicit function theorem, and actually you can directly find (*), skipping all the previous steps. After a little manipulation, this yields the desired relation:

$$\left(\frac{\partial z}{\partial x}\right)_{t,y,p} = -\left(\frac{\partial p}{\partial x}\right)_{t,y,z} \Big/ \left(-\rho g\right), \qquad g\left(\frac{\partial z}{\partial x}\right)_{t,y,p} = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{t,y,z},$$

$$\left(\frac{\partial \varphi}{\partial x}\right)_{t,y,p} = \alpha \left(\frac{\partial p}{\partial x}\right)_{t,y,z} .$$

We can repeat the same procedure to calculate the y derivative.

In summary,

$$\alpha\left(\left.\frac{\partial}{\partial x}\right|_{t,y,z}, \left.\frac{\partial}{\partial y}\right|_{t,x,z}\right) p = \left(\left.\frac{\partial}{\partial x}\right|_{t,y,p}, \left.\frac{\partial}{\partial y}\right|_{t,x,p}\right) \varphi \quad \text{or}$$

$$\alpha \nabla_z p = \nabla_p \varphi .$$