Quasi-Balanced Circulations in

Oceans and Atmospheres

Course Notes

2004

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1. Introduction

Most of the kinetic energy in atmospheric and oceanic circulations is tied up in flows that are dominated by rotational effects and that evolve slowly compared to a pendulum day (the time it takes a pendulum to complete one circuit, which is one sidereal day at 30° latitude). These include all the major ocean currents, from the thermohaline circulation to mesoscale ocean eddies; and the major atmospheric circulation systems ranging from synoptic-scale disturbances to planetary wave and other major elements of the general circulation. For the purposes of this course, we mean to distinguish circulations such as these from comparatively fast circulations such as internal inertia-gravity waves, and sound waves, convection, and three-dimensional turbulence. As we shall see, there is strong evidence that there does *not* exist an exact means of separating slow, rotationally dominated circulations from the faster ones; nor can it be assumed that there is no interaction between them. Nevertheless, it is usually possible to distinguish between observations of these two classes of circulation. In the course, we use the term *quasi-balanced* to refer to flows in which most of the kinetic energy is tied up in motions for which the hydrostatic approximation is valid and in which horizontal pressure gradients are nearly balanced by centrifugal accelerations. A more formal working definition of quasi-balanced flows is *those flows most of whose salient characteristics can be derived from the instantaneous potential vorticity distribution together with certain balance approximations and boundary conditions.* Even this definition lacks rigor for reasons we shall explore in due course, but it serves our purpose admirably in many ways, most especially as it stresses the central importance of *potential vorticity* in the description of the dynamics of quasi-balance flows.

The first part of this course will develop the *conservation* and *invertibility* principles that we will then use in the second part to describe the dynamics of quasi-balanced flows in the atmosphere and oceans. In doing so, we assume that you have taken a graduate-level course in geophysical fluid dynamics that covers the fundamental fluid laws at the level, say, of Joseph Pedlosky's book on the subject. You will find that the text entitled *Atmosphere-Ocean Dynamics,* by Adrian Gill (Academic Press, 1982) makes a nice companion to this course, though we will not by any means cover the same material in the same order. An essential reference is "On the use and significance of isentropic potential vorticity maps" by B. F. Hoskins, M. E. McIntyre, and A. W. Robertson, published in the *Quarterly Journal of the Royal Meteorological Society,* **111,** 1985, 877–946.

We begin with an elementary review of basic principles, followed by a derivation of various sets of filtered equations that will be used in the remainder of the course.

2. Fundamental balance and conservations principles for quasi-balanced flow

A. Hydrostatic balance

We shall assume throughout the course that quasi-balanced flows are very nearly hydrostatic. In local Cartesian coordinates, the vertical momentum equation is

$$\frac{dw}{dt} = -\alpha \frac{\partial p}{\partial z} - g + 2\Omega u \cos\varphi + \frac{u^2 + v^2}{a} + F_z, \tag{2.1}$$

where w is the vertical velocity, α is the specific volume, p is pressure, z is the upward vertical distance, g is the *effective* acceleration of gravity (which includes centripetal terms owing to the Earth's rotation), Ω is the angular velocity of the Earth's rotation, u is the zonal velocity, φ is the latitude, v is the meridional velocity, a is the mean radius of the earth and F_z is the vertical component of the acceleration owing to friction. The hydrostatic approximation is valid when the particle acceleration, Coriolis acceleration, and friction are all small compared to gravity:

$$\alpha \frac{\partial p}{\partial z} \simeq -g. \tag{2.2}$$

1. Application to the atmosphere

The atmosphere is well approximated by an ideal gas, whose specific volume is related to temperature, pressure, and water substance by

$$\alpha = \frac{R_d T_v}{p}, \tag{2.3}$$

where R_d is the gas constant of dry air, p is the total pressure, and T_v is the *virtual temperature,* defined

$$T_v = T\left(\frac{1 + r/\epsilon}{1 + r_t}\right), \tag{2.4}$$

where T is the absolute temperature, r is the mass mixing ratio of water vapor, and r_t is the mass mixing ratio of all water substance. The mass mixing ratios are defined as the mass of substance per unit mass of air exclusive of all water substance. The total water mixing ratio, r_t, includes condensed as well as vapor-phase water. The quantity ϵ is the ratio of the molecular weight of water to the mean molecular weight of dry air and has a value of 0.622.

Substituting (2.3) into (2.2) gives

$$R_d T_v \frac{\partial \ln p}{\partial z} = -g. \tag{2.5}$$

Integrating this results in

$$p = p_0 \exp\left[\frac{-g_0 z_g}{R_d \overline{T}_v}\right], \tag{2.6}$$

where g_0 is a standard value of g and

$$z_g \equiv \frac{1}{g_0}\int_0^z g\, dz$$

is the *geopotential height.* In the troposphere, the fractional change of g with altitude is small and so z_g is nearly equal to z. *For the purposes of this course, we will always use z to mean geopotential height and take g as the acceleration of gravity at sea level.*

In (2.6), $\overline{T}_v$ is the mean virtual temperature, defined

$$\overline{T}_v \equiv \frac{1}{\ln \frac{p_0}{p}} \int_p^{p_0} T_v \frac{dp}{p}. \tag{2.7}$$

If the virtual temperature is constant with pressure, as is nearly true in the lower stratosphere, then (2.6) shows that pressure decreases exponentially with altitude.

In the atmosphere, it is common to use *pressure* as the *independent vertical coordinate,* rather than altitude. In this coordinate system, (2.2) is usually written

$$\frac{\partial \varphi}{\partial p} = -\alpha, \tag{2.8}$$

where φ is the *geopotential,* defined

$$\varphi \equiv \int_0^z g\, dz, \tag{2.9}$$

where g is in this instance the full effective gravitational acceleration. Using (2.3) and (2.7), (2.8) can be integrated to yield

$$\varphi = R_d \overline{T}_v \ln \frac{p_0}{p}. \tag{2.10}$$

Also, the physical distance between two fixed pressure surfaces, often referred to as the *thickness,* is

$$\Delta z_g = \frac{R_d \tilde{T}_v}{g} \ln \frac{p_2}{p_1}, \tag{2.11}$$

where in this case

$$\tilde{T}_v \equiv \frac{1}{\ln \frac{p_2}{p_1}} \int_{p_1}^{p_2} T_v \frac{dp}{p}. \tag{2.12}$$

2. Application to the ocean

A convenient density variable to use in the ocean is σ, defined

$$\sigma \equiv (\rho - 1) \times 10^3, \tag{2.13}$$

where ρ is the density in g cm^{-3}. In general, σ (or ρ) is a function of pressure, temperature, and salinity:

$$\sigma = \sigma(s, T, p).$$

The equation of state for sea water is not as simple as its atmospheric counterpart. It may be written approximately as

$$\alpha = \frac{c_1 + c_2 T + c_3 T^2 - c_4 S - c_5 TS}{p + c_6 + c_7 T - c_8 T^2 + c_9 S}, \tag{2.14}$$

with α in cm^3 g^{-1}, p in bars, T in °C, and S in 0/00 (grams of dissolved substance per kilogram of sea water). The constants in (2.14) are

$$c_1 = 1752.73,$$

$$c_2 = 11.01,$$

$$c_3 = 0.0639,$$

$$c_4 = 3.9986,$$

$$c_5 = 0.0107$$

$$c_6 = 5880.9,$$

$$c_7 = 37.592,$$

$$c_8 = 0.34395,$$

$$c_9 = 2.2524.$$

In the ocean, it is conventional to define z as *positive downward,* so the hydrostatic equation may be written

$$\frac{\partial p}{\partial z} = \rho g. \tag{2.15}$$

Since ρ is nearly constant, the vertical pressure gradient in the ocean is nearly equal to 1 db m^{-1}, where *db* stands for *decibar.*

3. Equations of motion

For the purposes of this course, we will, for the most part, use the hydrostatic, horizontal equations of motion in local Cartesian coordinates. The full equations may be written (cf. Holton, 1992)

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = -\alpha\frac{\partial p}{\partial x} + 2\Omega\sin\varphi v - 2\Omega\cos\varphi w + F_x, \tag{3.1}$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -\alpha\frac{\partial p}{\partial y} - 2\Omega\sin\varphi u + F_y, \tag{3.2}$$

where u and v are the eastward and northward velocity components, and F_x and F_y are the components of frictional acceleration in the eastward and northward directions.

A scale analysis of these momentum equations (cf. Holton, 1992) shows that the centrifugal terms on the left sides of (3.1) and (3.2) are very small compared to the other terms, as is the Coriolis acceleration term involving w in (3.1). Thus, for the purposes of forming simplified equations for developing conceptual understanding of quasi-balanced flows, we shall drop these terms henceforth, and write (3.1) and (3.2) as

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x} + fv + F_x, \tag{3.3}$$

$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial y} - fu + F_y, \tag{3.4}$$

where f is the *Coriolis parameter,* defined

$$f \equiv 2\Omega \sin\varphi. \tag{3.5}$$

In the atmosphere, it is usually convenient to write (3.3) and (3.4) in pressure coordinates:

$$\frac{du}{dt} = -\frac{\partial \varphi}{\partial x} + fv + F_x, \tag{3.6}$$

$$\frac{dv}{dt} = -\frac{\partial \varphi}{\partial y} - fu + F_y, \tag{3.7}$$

in which horizontal gradients are understood to be taken at constant pressure.

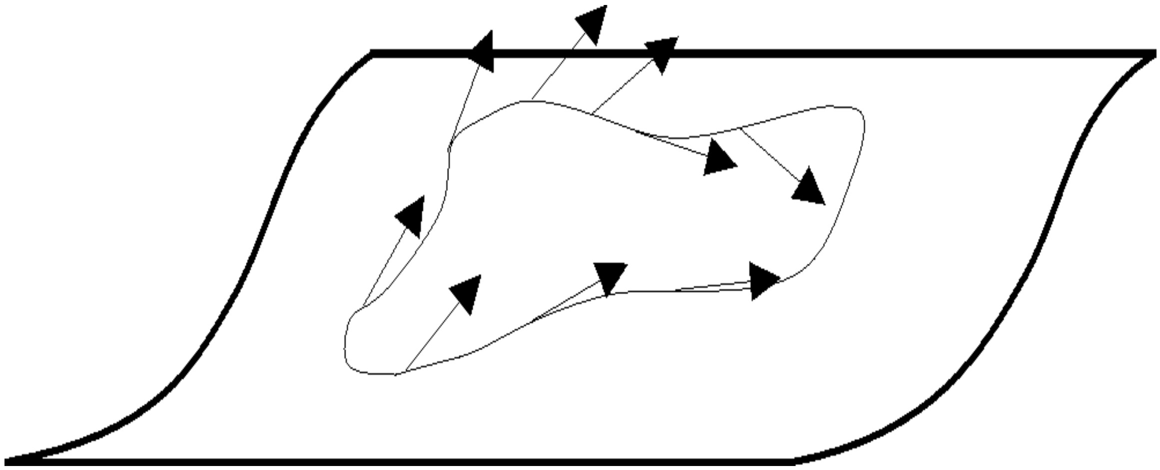

Figure 3.1

5. Circulation

A key integral conservation property of fluids is the *circulation.* For convenience, we derive *Kelvin's circulation theorem* in an inertial coordinate system and later transform back to Earth coordinates.

In inertial coordinates, the vector form of the momentum equation may be written

$$\frac{d\mathbf{V}}{dt} = -\alpha\nabla p - g\hat{k} + \mathbf{F}, \tag{4.1}$$

where $\hat{k}$ is the unit vector in the z direction and $\mathbf{F}$ represents the vector frictional acceleration.

Now define a material curve that, however, is constrained to lie at all times on a surface along which some *state variable,* which we shall refer to for now as s, is a constant. (A state variable is a variable that can be expressed as a function of temperature and pressure.) The picture we have in mind is shown in Figure 3.1.

The arrows represent the projection of the velocity vector onto the s surface, and the curve is a *material curve* in the specific sense that each point on the curve moves with the vector velocity, projected onto the s surface, at that point.

Now the *circulation* is defined as

$$C \equiv \oint \mathbf{V} \cdot d\mathbf{l}, \tag{4.2}$$

where $d\mathbf{l}$ is an incremental length *along* the curve and $\mathbf{V}$ is the vector velocity. The integral is a closed integral around the curve. Differentiation of (4.2) with respect to this gives

$$\frac{dC}{dt} = \oint \frac{d\mathbf{V}}{dt} \cdot d\mathbf{l} + \oint \mathbf{V} \cdot \frac{d\mathbf{l}}{dt}. \tag{4.3}$$

Since the curve is a material curve,

$$\frac{d\mathbf{l}}{dt} = d\mathbf{V},$$

so the integrand of the last term in (4.3) can be written as a perfect derivative and so the term itself vanishes. Thus

$$\frac{dC}{dt} = \oint \frac{d\mathbf{V}}{dt} \cdot d\mathbf{l},$$

and substituting (4.1) results in

$$\frac{dC}{dt} = \oint \left[-\alpha \nabla p + \mathbf{F}\right] \cdot d\mathbf{l}. \tag{4.4}$$

The gravity term vanishes because it can be expressed as the derivative of a potential, and so vanishes when integrated on a closed curve.

Now since α is a state variable (neglecting its dependence on water vapor or, in the ocean, salinity, for the time being), it can be written as a function of s and p:

$$\alpha = \alpha(s, p),$$

But because ∇p in (4.4) must be a gradient at *constant* s (since the material curve is chosen to lie on an s surface), the pressure term in (4.4) can be written

$$-\oint \alpha \nabla p \cdot dl = \int \nabla \zeta \cdot dl = 0,$$

where

$$\zeta \equiv \int_0^p \alpha(s_0, p) dp',$$

and s_0 is the particular value of s characterizing the s surface in question. Thus (4.4) becomes

$$\frac{dC}{dt} = \oint \mathbf{F} \cdot d\mathbf{l}. \tag{4.5}$$

Thus the only process that changes the circulation around a closed material curve on an s surface is friction.

Using the definition of circulation, (4.2), and Stokes's theorem, (4.5) can be written alternatively as

$$\frac{d}{dt} \int \int_A (\nabla \times \mathbf{V}) \cdot \hat{n} dA = \int \int_A (\nabla \times F) \cdot \hat{n} dA, \tag{4.6}$$

where now the integrals are over the area enclosed by the material curve, and $\hat{n}$ is a unit vector orthogonal to the s surface.

In either of its two forms, (4.5) or (4.6), the Circulation Theorem expresses a fluid analog of angular momentum conservation. If the curve happens to be a circle, then the circulation can be seen to be the angular momentum per unit mass of an infinitesimal ring centered on that circle (multiplied by 2π).

Now in a local coordinate system fixed to the rotating earth, the absolute velocity (which appears in (4.5) and (4.6)) is related to the Earth-relative velocity, $\mathbf{V}_r$, by

$$\nabla \times \mathbf{V} = \nabla \times \mathbf{V}_r + 2\mathbf{\Omega}, \tag{4.7}$$

where $\mathbf{\Omega}$ is the vector angular velocity of the Earth's rotation. Substituting (4.7) into (4.6) gives

$$\frac{d}{dt}\int\int [\nabla \times \mathbf{V}_r + 2\mathbf{\Omega}] \cdot \hat{n} dA = \int\int (\nabla \times \mathbf{F}) \cdot \hat{n} dA. \tag{4.8}$$

This is the form of the Circulation Theorem that we shall most often refer to. It implies that on a surface along which some state variable is constant, the relative vorticity ($\nabla \times \mathbf{V}_r$) will *increase* if

1. The area enclosed by the material curve decreases (implying convergence).
2. The curve is displaced southward or is tilted, such that $\mathbf{\Omega} \cdot \hat{n}$ decreases.

Henceforth, we shall assume Earth-relative coordinates and drop the subscript r in (4.8).

It is worth noting that the dependence of specific volume, α, on water vapor in the atmosphere and on salinity in the ocean can be accounted for in forming the circulation theorem, by a suitable choice of variables. In the atmosphere, a good choice of variables is the *virtual potential temperature,* defined

$$\theta_v \equiv T_v \left(\frac{p_0}{p}\right)^{\kappa}, \tag{4.9}$$

where T_v is given by (2.4), p_0 is a reference pressure (usually 1000 MBA), and κ is R_d/c_{pd}, where R_d is the gas constant for dry air, and c_{pd} is the heat capacity at constant pressure of dry air. It can be shown that θ_v is very nearly conserved in reversible adiabatic transformations. Moreover, as is clear from (2.3) and (4.9),

$$\alpha = \frac{R_d T_V}{p} = R_d \theta_v p^{\kappa-1} p_0^{-\kappa}, \tag{4.10}$$

and on a surface along which θ_v is constant,

$$\alpha \nabla p = c_{pd} \theta_v p_0^{-\kappa} \nabla p^{\kappa},$$

so that once again,

$$\oint \alpha \nabla p \cdot d\mathbf{l} = 0,$$

when the curve lies on a surface of constant θ_v.

Similarly, in the ocean, we can write

$$\alpha = \sigma(s, T, p) G(p), \tag{4.11}$$

where σ is the *potential density* and is a function of salinity, temperature, and pressure. The potential density is the density sea water would have if brought reversibly to some reference pressure. Clearly, from (4.11), the pressure gradient term also vanishes when integrated around a closed material curve on a surface along which σ is constant.

While the Circulation Theorem is useful for envisioning changes in fluid vorticity, it is an integral theorem and thus cannot be used as part of a closed system of equations describing the detailed evolution of fluid flow. We therefore proceed to develop a conserved scalar, called the potential vorticity, using the Circulation Theorem as a starting point.

5. Potential vorticity

Consider two closed material curves on two adjacent surfaces along which some state variable (e.g. σ or θ_v) is a constant, as illustrated in Figure 5.1. We will suppose that the area enclosed by the curves is infinitesimal, as is the distance between the two s surfaces, and that the curves are connected by material walls so as to form a material volume, which is material only in the sense that points on its surface move with the component of the total fluid velocity projected onto s surfaces, as with the derivation of the Circulation Theorem. Thus, material may leave or enter the *ends* of the material volume, but not the sides.

The amount of mass contained in the volume is

$$\delta M = \rho\, \delta A\, \delta n, \tag{5.1}$$

where ρ is the fluid density and δn is the distance between adjacent s surfaces.

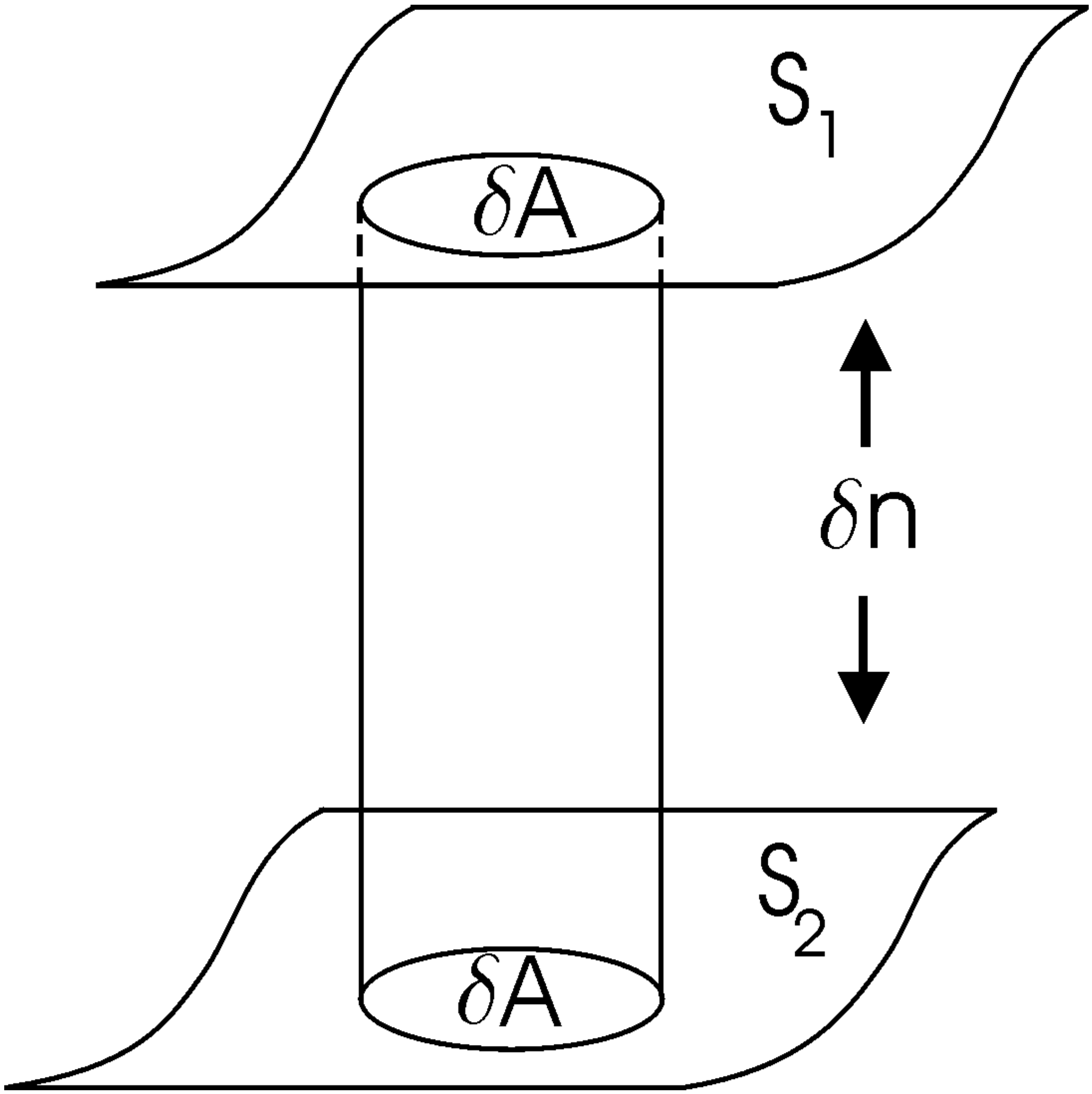

Figure 5.1

Since n lies in the direction of ∇s,

$$\delta n = \frac{dn}{ds}\delta s, \tag{5.2}$$

where $\delta s = s_2 - s_1$ is the difference between the values of s on the two surfaces.

Using (5.1) and (5.2), the incremental area, δA, may be written

$$\delta A = \frac{1}{\rho \delta s}\frac{ds}{dn}\delta M. \tag{5.3}$$

Using this, the Circulation Theorem in the form (4.8), with the integrals taken over the infinitesimal areas, can be written

$$\frac{d}{dt}\left[(\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \cdot \hat{n} \frac{1}{\rho \delta s}\frac{ds}{dn}\delta M\right] = (\nabla \times \mathbf{F}) \cdot \hat{n} \frac{1}{\rho \delta s}\frac{ds}{dn}\delta M. \tag{5.4}$$

Given that the direction of $\hat{n}$ is the same as that of ∇s, and that δs is by definition a fixed increment in s, (5.4) may be re-expressed

$$\frac{d}{dt}\left[\alpha(\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \cdot \nabla s \, \delta M\right] = \alpha(\nabla \times \mathbf{F}) \cdot \nabla s \, \delta M. \tag{5.5}$$

The variability of δM can now be related to sources and sinks of s, since clearly if fluid is entering or leaving through the ends of the volume, which lie on surfaces of constant s, then there must be sources or sinks of s.

The rate of mass flow across each end of the cylinder is

$$\rho \frac{ds}{dt}\frac{dn}{ds}\delta A,$$

so that the rate of change of mass in the cylinder is the convergence of the flux:

$$\frac{d}{dt}\delta M = -\frac{d}{dn}\left[\rho \frac{ds}{dt}\frac{dn}{ds}\delta A \, \delta n\right] = -\rho \delta A \, \delta n \frac{\partial}{\partial s}\frac{ds}{dt} = -\delta M \frac{\partial}{\partial s}\frac{ds}{dt}. \tag{5.6}$$

Using this in (5.5) gives

$$\boxed{\begin{aligned} &\frac{d}{dt}\left[\alpha(\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \cdot \nabla s\right] = \\ &\quad \alpha(\nabla \times \mathbf{F}) \cdot \nabla s + \alpha(\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \cdot \nabla \frac{SD}{dt}. \end{aligned}} \tag{5.7}$$

This known at *Ertel's Theorem* and states that the quantity

$$\alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla s$$

is conserved following the fluid flow, in the absence of friction or sources or sinks of s.

It is customary to use θ as the relevant state variable in the atmosphere, although it is more accurate to use θ_v, since it accounts for the dependence of density of water vapor. We shall therefore define the *potential vorticity,* for atmospheric applications, as

$$\boxed{q_a \equiv \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\theta_v,} \tag{5.8}$$

and in the ocean, we will use potential density, σ, for s:

$$\boxed{q_o \equiv \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\sigma.} \tag{5.9}$$

Thus, according to (5.7), the conservation equations for q_a and q_o are

$$\frac{dq_a}{dt} = \alpha(\nabla \times \mathbf{F}) \cdot \nabla\theta_v + \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\frac{d\theta_v}{dt}, \tag{5.10}$$

and

$$\frac{dq_o}{dt} = \alpha(\nabla \times \mathbf{F}) \cdot \nabla\sigma + \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \frac{d\sigma}{dt}. \tag{5.11}$$

In the atmosphere, potential vorticity is conserved in the absence of friction or sources of θ_v; it is conserved in the ocean in the absence of friction or sources of σ.

Returning to Figure 5.1, it is seen that in the absence of sources or sinks of s, the drawing together of the two s surfaces implies, by mass conservation, that the volume expands laterally as it is squashed; by the Circulation Theorem, the absolute vorticity must decrease. This is precisely what (5.7) indicates. Potential vorticity can be thought of as that vorticity a fluid column would have if it were stretched or squashed to some reference depth.

Volume conservation of potential vorticity

The integral of potential vorticity over a finite mass of fluid is conserved even in the presence of friction or sources of σ or θ_v, as long as those effects vanish at the *boundaries* of that mass of fluid. The potential vorticity tendency integrated over a fixed (material) mass is

$$\int\int\int \frac{dq}{dt}\rho\, dx\, dy\, dz = \frac{d}{dt}\int\int\int q\rho\, dx\, dy\, dz.$$

We can take the time derivative outside the integral because mass is conserved. Using (5.7)

$$\frac{d}{dt}\int\int\int \rho q\, dx\, dy\, dz = \int\int\int \left[(\nabla \times \mathbf{F}) \cdot \nabla s + (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla \frac{ds}{dt}\right] dx\, dy\, dz. \tag{5.12}$$

Since the divergence of the curl of any vector vanishes, and since $\mathbf{\Omega}$ is a constant vector, (5.12) can be rewritten

$$\frac{d}{dt}\int\int\int \rho q\,dx\,dy\,dz = \int\int\int \nabla\cdot\left[s(\nabla\times\mathbf{F}) + \frac{ds}{dt}(\nabla\times\mathbf{V}+2\mathbf{\Omega})\right]dx\,dy\,dz$$
$$= \int\int\left[s(\nabla\times\mathbf{F}) + \frac{ds}{dt}(\nabla\times\mathbf{V}+2\mathbf{\Omega})\right]\cdot\hat{n}\,dA, \tag{5.13}$$

where the last integral is over the entire surface bounding the volume and $\hat{n}$ is a unit vector normal to that surface. (We have used the divergence theorem here.) Thus, *the mass integral of q is conserved if there are no sources of s or friction on the boundaries of the fluid mass.*

6. Invertibility

The potential vorticity, q, is in general a function of the distributions of 5 variables: the three velocity components, density, and either θ_v or σ. In *quasi-balanced* flows, it is possible to reduce this dependence to one that relies on a *single* variable, from which all of the others can be derived. The relationship between q and this single variable is usually through an elliptic, second-order differential equation. Under these circumstances, the spatial distribution of q can be *inverted,* given certain boundary conditions, to yield the distribution of velocity and mass. This property of q and of quasi-balanced flows is known as *invertibility.*

We shall explore various balance approximations in some detail later, but now let's have a quick look at how the dependence of q on 5 variables may be reduced to a dependence on 1 under some conditions.

First, let's expand the definition of potential vorticity out into its various components. For the atmosphere, (5.8) expands to

$$q_a = \alpha \left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta_v}{\partial z} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \frac{\partial \theta_v}{\partial y} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \frac{\partial \theta_v}{\partial x} \right]. \tag{6.1}$$

From mass continuity, w scales at *most* according to

$$\frac{\partial w}{\partial z} \sim 0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

or

$$w \sim 0 \left(u_0 \frac{H}{L} \right), \tag{6.2}$$

where H and L are typical vertical and horizontal scales over which the flow varies, and u_0 is a typical horizontal velocity scale. (Note that in most geophysical flows, the flow is quasi-nondivergent, so actually $w \ll 0 \left(u_0 \frac{H}{L} \right)$.) Thus, in terms that appear in (6.1), like

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

the order of the term is

$$\frac{u_0}{H} \left(1 - \frac{H^2}{L^2} \right).$$

Since, for virtually all flows we will be interested in, $H/L \ll 1$, the contribution of w to the potential vorticity is utterly negligible. So (6.1) may be accurately approximated by

$$q_a \simeq \alpha \left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta_v}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial \theta_v}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial \theta_v}{\partial x} \right]. \tag{6.3}$$

Now if we employ the hydrostatic approximation, (2.2), it follows that

$$\alpha \frac{\partial A}{\partial z} \simeq -g \frac{\partial A}{\partial p},$$

for any quantity A. Using this in (6.3) gives

$$q_a \simeq -g\left[\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial\theta_v}{\partial p}+\frac{\partial u}{\partial p}\frac{\partial\theta_v}{\partial y}-\frac{\partial v}{\partial p}\frac{\partial\theta_v}{\partial x}\right], \tag{6.4}$$

This can be re-expressed *in θ_v coordinates* as

$$q_a \simeq -g\left(\frac{\partial p}{\partial\theta_v}\right)^{-1}\left(f+\left(\frac{\partial v}{\partial x}\right)_{\theta_v}-\left(\frac{\partial u}{\partial y}\right)_{\theta_v}\right). \tag{6.5}$$

Now suppose that the flow is, to a good approximation, hydrostatic and geostropheICC. In θ_v coordinates, the hydrostatic and geostrophic relations are expressed in terms of the *Montgomery streamfunction:*

$$M \equiv c_{pd}T_v + gz. \tag{6.6}$$

These relations are:

Hydrostatic:

$$c_{pd}\left(\frac{p}{p_0}\right)^{\kappa}=\frac{\partial M}{\partial\theta_v}, \tag{6.7}$$

Geostrophic:

$$\begin{aligned} fu_g &= -\left(\frac{\partial M}{\partial y}\right)_{\theta_v}, \\ fv_g &= \left(\frac{\partial M}{\partial x}\right)_{\theta_v}. \end{aligned} \tag{6.8}$$

Substituting these into (6.5) gives

$$q_a \simeq -gp_0^{-1}c_{pd}^{\frac{1}{\kappa}}\kappa\left(f+\frac{1}{f}\nabla^2 M\right)\left[\left(\frac{\partial M}{\partial\theta_v}\right)^{\frac{1}{\kappa}-1}\frac{\partial^2 M}{\partial\theta_v^2}\right]^{-1}. \tag{6.9}$$

Then q_a is a function of M alone, and this function is a nonlinear and usually elliptic one. (It is always elliptic when $\frac{1}{f}\nabla^2 M + f$ has the same sign as q_a, and

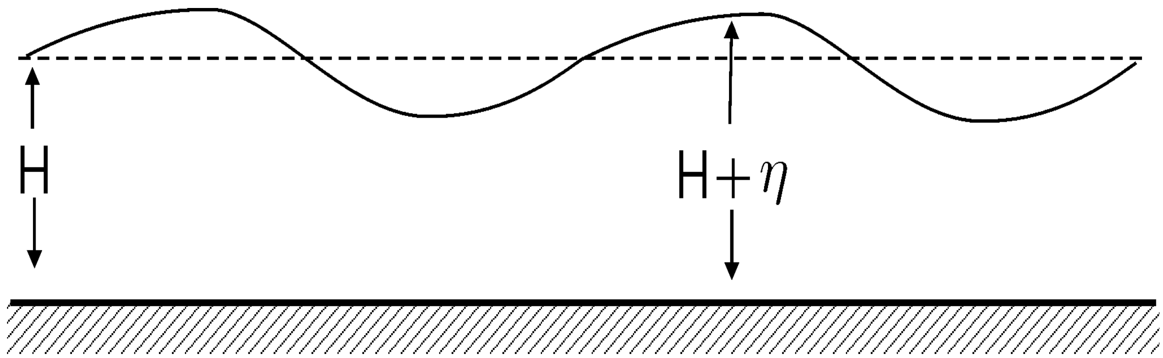

Figure 7.1

$\partial M/\partial\theta_v > 0$.) When it is elliptic, (6.9) can be inverted to find M, and therefore u_g, v_g, and p, given the distribution of q_a and certain boundary conditions.

We will be developing somewhat simpler invertibility relationships for potential vorticity. The essential elements in all of these are the definition of potential vorticity, coupled with balance approximations that link the instantaneous distribution of velocity to that of mass.

7. Potential vorticity and invertibility in the shallow water equations

Consider a layer of strictly incompressible fluid on a level surface, as illustrated in Figure 7.1. The mean depth of the fluid is H with perturbations to the depth denoted by η.

Since the fluid is incompressible, mass continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7.1}$$

We shall assume that all the motions of interest are hydrostatic, so that integration of the hydrostatic equation gives

$$p = p_0 + \rho g(H + \eta - z), \tag{7.2}$$

where p_0 is the atmospheric pressure and ρ is the fluid density. At the surface of the fluid,

$$w = \frac{d\eta}{dt}. \tag{7.3}$$

The inviscid momentum equations may be written, with the aid of (7.2), as

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} + fv, \tag{7.4}$$

$$\frac{dv}{dt} = -g\frac{\partial \eta}{\partial y} - fu. \tag{7.5}$$

Note from (7.2) that the horizontal pressure gradient acceleration is *independent of depth* in the fluid, as reflected in (7.4) and (7.5). So (7.4) and (7.5) show that if at the initial time, u and v are independent of depth, they will remain so forever. *We shall assume that u and v are depth-independent.* This means that the mass continuity equation, (7.1), can be integrated over the local depth of the fluid to give

$$(H + \eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{d\eta}{dt} = 0, \tag{7.6}$$

in which we have applied (7.3).

Given that u and v are independent of depth, (7.4) and (7.5) may be expressed in the alternative form

$$\frac{\partial u}{\partial t} - (f + \zeta)v = -\frac{\partial B}{\partial x}, \tag{7.7}$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = -\frac{\partial B}{\partial y}, \tag{7.8}$$

where ζ is the relative vorticity,

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{7.9}$$

and B is the Bernoulli function:

$$B \equiv g\eta + \frac{1}{2}(u^2 + v^2). \tag{7.10}$$

If B is eliminated by cross differentiating (7.7) and (7.8), the result is a vorticity equation:

$$\frac{d\zeta}{dt} + (f + \zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0, \tag{7.11}$$

showing that vertical stretching is the only source of vorticity in the shallow water system.

Now the horizontal divergence,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

may be eliminated between (7.6) and (7.11) to arrive at the *shallow water potential vorticity equation:*

$$\frac{dq}{dt} = 0, \tag{7.12}$$

with

$$q \equiv \frac{f + \zeta}{H + \eta}. \tag{7.13}$$

If the flow velocity and η are related by some condition, then clearly (7.12), (7.13) constitute a closed system.

One example in which q and $\mathbf{v}$ are strictly related to one another is *steady flow.* First, note that the mass continuity equation, (7.6), can be written in the alternative form

$$\frac{\partial}{\partial x}\left[u(H+\eta)\right]+\frac{\partial}{\partial y}\left[v(H+\eta)\right]=0, \tag{7.14}$$

in the case of steady flow. Then we may define a *mass streamfunction,* ψ, such that

$$\begin{aligned} u(H+\eta) &\equiv -\frac{\partial\psi}{\partial y}, \\ v(H+\eta) &\equiv \frac{\partial\psi}{\partial x}, \end{aligned} \tag{7.15}$$

which clearly satisfies (7.14).

Now the steady forms of the momentum equations, (7.7) and (7.8), are

$$(f+\zeta)v=\frac{\partial B}{\partial x}, \tag{7.16}$$

$$(f+\zeta)u=-\frac{\partial B}{\partial y}, \tag{7.17}$$

or using (7.15) and (7.13),

$$q\frac{\partial\psi}{\partial x}=\frac{\partial B}{\partial x}, \tag{7.18}$$

$$q\frac{\partial\psi}{\partial y}=\frac{\partial B}{\partial y}. \tag{7.19}$$

At the same time, the steady form of the potential vorticity equation, (7.12), is

$$u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}=0. \tag{7.20}$$

Multiplying (7.20) by $H+\eta$ and using (7.15) gives

$$-\frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y}=0,$$

or

$$J(q, \psi) = 0, \tag{7.21}$$

where J is the Jacobian operator. The direct implication of (7.21) is that $q = q(\psi)$. Given this conclusion, it is clear from (7.18) and (7.19) that $B = B(\psi)$. Thus if $q(\psi)$ is *known,* then $B(\psi)$ can be found from (7.18) and (7.19), and (7.10) and (7.13) can then be solved for η, u, and v. This is an example of inversion, and for this case it is exact.

In time-dependent flows, it is necessary to make an approximation to the momentum equations to recover invertibility. *The most commonly used approximation in geophysical flows is to assume that the horizontal divergence, or its time tendency, is small compared to the vertical component of vorticity;* this approximation filters sound waves and inertia-gravity waves from the equations.

We start by differentiating (7.4) with respect to x and (7.5) with respect to y and summing the result to arrive at the *divergence equation:*

$$\frac{dD}{dt} = -g\nabla^2\eta + (f + \zeta)\zeta - \beta u - S^2, \tag{7.22}$$

where

$$\begin{aligned} D &\equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \\ \zeta &\equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \\ \beta &\equiv \frac{df}{dy}, \end{aligned}$$

and

$$S^2 \equiv \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 .$$

S is the net deformation in the flow.

The *nonlinear balance equation* is obtained by making the approximation

$$|\frac{dD}{dt}| \ll |(f+\zeta)\zeta|, \tag{7.23}$$

so that

$$\boxed{g\nabla^2\eta \simeq (f+\zeta)\zeta - \beta u - S^2.} \tag{7.24}$$

This relates the *instantaneous* distribution of η to the *instantaneous* distribution of velocity in the system.

Now to be consistent with the approximation (7.23), the velocities that appear on the right side of (7.24) should be dominated by their nondivergent components, allowing us to rewrite (7.24) as

$$g\nabla^2\eta \simeq (f+\nabla^2\psi)\nabla^2\psi + \beta\frac{\partial\psi}{\partial y} - 2\left(\frac{\partial^2\psi}{\partial y\partial x}\right)^2 - \left(\frac{\partial^2\psi}{\partial y^2}\right)^2 - \left(\frac{\partial^2\psi}{\partial x^2}\right)^2 , \tag{7.25}$$

where ψ is now the velocity streamfunction, defined such that

$$\begin{aligned} u &= -\frac{\partial\psi}{\partial y}, \\ v &= \frac{\partial\psi}{\partial x}. \end{aligned} \tag{7.26}$$

At the same time, the potential vorticity equation (7.12) with (7.13) can be written

$$\frac{\partial q}{\partial t} = -J(\psi, q). \tag{7.27}$$

The two equations (7.27) and (7.24) constitute a *closed system* for ψ and η. Potential vorticity is just advected around, according to (7.27), and at each time the streamfunction and depth η can be found from (7.25) and

$$q = \frac{f + \nabla^2\psi}{H + \eta}. \tag{7.28}$$

It should be mentioned here that the approximation (7.23) does *not* imply that $w = 0$ from mass continuity; it is simply a scaling approximation appropriate to the divergence equation. In fact, the system of equations (7.27), (7.28), and (7.25) directly imply a nonzero horizontal divergence, because in general the evolution of the system will give a nonzero charge in $f + \nabla^2\psi$, the vorticity. But from the vorticity equation, (7.11), vorticity can *only* change if there is nonzero divergence. Thus the divergence and the vertical velocity can be diagnosed by solving (7.11) once the time rate of change of vorticity is known (or by solving (7.6) once the time rate of change of η is known.) This is why the system is referred to as a *quasi-*balanced system: Exact balance is degenerate in the sense that the evolution of the flow cannot be calculated.

In summary, a suitable balance approximation allows one to calculate the evolution of a quasi-balanced flow according to this schema:

1. Prescribe initial flow and/or η that satisfies the balance condition (7.25, in this case).
2. Calculate the distribution of q using its definition (7.28).
3. Find q at the next time step by solving the potential vorticity equation, (7.27).

4. Invert the q distribution using the definition of q and a balance condition (7.28 and 7.25, in this case) to find the velocity and mass distribution. (Note, however, that boundary conditions must be prescribed to carry out the inversion.)
5. Go back to 3.

Note that this scheme is highly analogous to the solution of the vorticity equation for strictly two-dimensional flow:

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta), \tag{7.29}$$

$$\zeta = \nabla^2 \psi. \tag{7.30}$$

Here again, vorticity is simply advected around by the flow, and the flow is recovered at each time step by inverting (7.30) subject to boundary conditions. The difference is that the system (7.29), (7.30) for two-dimensional flow is *exact,* while quasi-balance systems like (7.25), (7.27), and (7.28) rely on a balance approximation like (7.23). But clearly, the evolution of quasi-balanced flows is strongly analogous to the evolution of two-dimensional flows.

Illustration of balance and invertibility principles: the Rossby adjustment problem

Consider a two-dimensional slab of incompressible fluid, which at time $t = 0$ has a rectangular cross section (Figure 7.2). The fluid is contained on an f plane ($f =$ constant). At time $t = 0$, the fluid is released and allowed to evolve in time. We will approximate the flow as inviscid. What happens?

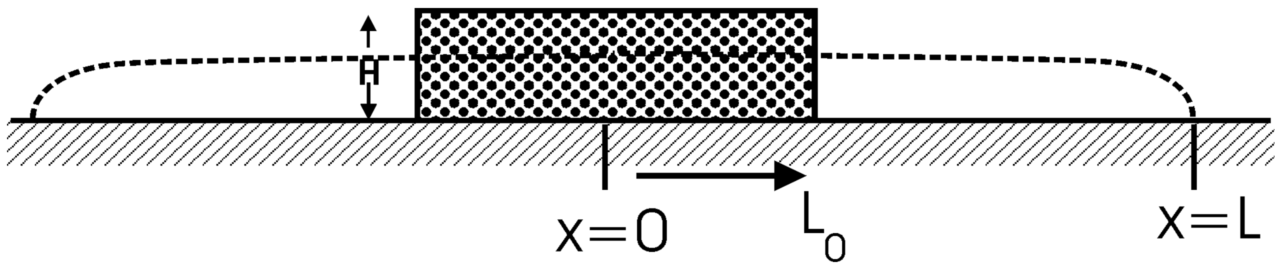

Figure 7.2

Clearly, the fluid will spread horizontally under the influence of gravity, acquiring velocities in the y direction through Coriolis accelerations. Gradually, the Coriolis accelerations acting on these velocities in the y direction will begin to balance the pressure gradient accelerations in the x direction.

We can find steady, two-dimensional solutions of the shallow water equations by assuming that potential vorticity is conserved and by solving the x-momentum equations for steady flow. This is a state toward which the system presumably evolves.

The potential vorticity, from (7.13), is

$$q = \frac{f + \frac{\partial v}{\partial x}}{\eta + H}, \tag{7.31}$$

where v is the velocity in the y direction. At $t = 0$, this is

$$q = \frac{f}{H}, \tag{7.32}$$

and since q is conserved, it must equal this value at all times. Equating (7.31) and (7.32) gives

$$\frac{\partial v}{\partial x} = f \frac{\eta}{H}. \tag{7.33}$$

The momentum equations in the x direction is

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} + fv. \tag{7.34}$$

In steady equilibrium,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0, \tag{7.35}$$

since the flow is steady and two-dimensional ($\partial/\partial y = 0$) and, we presume, $u = 0$. Thus the final state is geostrophically balanced:

$$g\frac{\partial \eta}{\partial x} = fv, \tag{7.36}$$

and eliminating v between (7.34) and (7.36) gives an ordinary differential equation for η:

$$\frac{g}{f}\frac{d^2\eta}{dx^2} - \frac{f}{H}\eta = 0. \tag{7.37}$$

The solution of this equation that satisfies the condition $H + \eta = 0$ at $x = \pm L$ (where L is, by definition, the distance from the symmetry axis to which the fluid spreads) is

$$\eta = -H\frac{\cosh\frac{x}{L_D}}{\cosh\frac{L}{L_D}}, \tag{7.38}$$

where L_D is the *deformation radius,* defined

$$L_D \equiv \frac{\sqrt{gH}}{f}. \tag{7.39}$$

This is an important scale in all quasi-balanced flows.

Now we can find the scale L to which the fluid spreads by demanding that mass be conserved:

$$\int_0^L (H+\eta)dx = HL_0. \tag{7.40}$$

The right-hand side of (7.40) is the half-volume of the initial state, while the left side is the half-volume of the final state.

Performing the integration in (7.40) after substituting (7.38) gives a transcendental equation for L:

$$\frac{L}{L_D} - \tanh\left(\frac{L}{L_D}\right) = \frac{L_0}{L_D}. \tag{7.41}$$

Given L_0, this allows us to calculate L. Note that all length scales are now effectively normalized by the deformation radius.

It is instructive to look at solutions to (7.41) in certain limiting cases:

1. $L_0/L_D \gg 1$: The initial horizontal scale is "large." Since $L > L_0$, and $\lim_{x\to\infty} \tanh x = 1$, (7.41) becomes

$$\frac{L}{L_D} - 1 \to \frac{L_0}{L_D},$$

or

$$\boxed{L \to L_0 + L_D.} \tag{7.42}$$

The fluid spreads approximately one deformation radius beyond its initial scale.

2. $L_0/L_D \ll 1$: The initial horizontal scale is "small." In this case, if we assume that $L/L_0 \ll 1$, the tanh term in (7.41), expanded to second order in L/L_D, is

$$\tanh \frac{L}{L_D} = \frac{L}{L_D} - \frac{1}{3}\left(\frac{L}{L_D}\right)^3 + O\left(\frac{L}{L_D}\right)^5$$

and, therefore,

$$\boxed{L \to (3L_0)^{1/3} L_D^{2/3}.} \tag{7.43}$$

The fluid spreads to a scale that represents a geometric average of the deformation radius and the initial scale, weighted toward the former.

It is worth noting that when $L_0 = L_D$, the solution to (7.41) is $L = 1.96 L_D$, which is much closer to the large-scale asymptotic limit given by (7.42).

This example is one of *geostrophic adjustment* in which an initially highly unbalanced flow adjusts to a balanced state. Note that in the large-scale limit, the mass distribution hardly changes at all, but the v velocity component undergoes a large adjustment near the edges of the block. On the other hand, in the limit of a small-scale initial length, the mass field undergoes a large adjustment while the velocities remain small.

8. Quasigeostrophy and Pseudo-potential vorticity

The shallow water system is probably the most simple fluid system that allows for divergent flow and inertia-gravity waves. Here we develop a simple set of equations for quasi-balanced flows of a *continuously stratified fluid,* based on the approximation that the flow is nearly geostrophic. This system is called the *quasi-geostrophic system.*

We begin with the horizontal momentum equation in pressure coordinates:

$$\frac{d\mathbf{V}}{dt} + f\hat{k} \times \mathbf{V} + \nabla\varphi = \mathbf{F}, \tag{8.1}$$

where $\mathbf{F}$ is the net acceleration by frictional forces. *Geostrophic balance* is defined by the equality of the two middle terms of (8.1), so that the *geostrophic wind* is defined

$$\mathbf{V}_g \equiv \frac{1}{f}\hat{k} \times \nabla\varphi. \tag{8.2}$$

Using this definition, (8.1) may be rewritten

$$\frac{d\mathbf{V}}{dt} + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} = 0, \tag{8.3}$$

or, equivalently,

$$\mathbf{V} = \mathbf{V}_g + \frac{1}{f}\hat{k} \times \frac{d\mathbf{V}}{dt} - \frac{1}{f}\hat{k} \times \mathbf{F}. \tag{8.4}$$

We will use (8.4) to investigate the relative magnitudes of the terms in the horizontal momentum equations. For this purpose, we shall approximate frictional acceleration as

$$\mathbf{F} \simeq -\mathbf{V}/\tau_f, \tag{8.5}$$

where τ_f is a time scale associated with frictional damping. We also define a *Lagrangian time scale,* $\boldsymbol{\tau}$, which can be thought of as a typical time scale over which a sample of fluid accelerates in a given flow. We replace the dimensional time, t, in (8.4) by a nondimensional time t^*:

$$t \rightarrow \tau t^*, \tag{8.6}$$

resulting in the scaled version of (8.4):

$$\mathbf{V} = \mathbf{V}_g + R_0 \hat{k} \times \frac{d\mathbf{V}}{dt} + R_F \hat{k} \times \mathbf{V}, \tag{8.7}$$

where R_0 is the *Rossby number,* defined

$$R_0 \equiv \frac{1}{f\tau}, \tag{8.8}$$

and R_F is a nondimensional measure of friction:

$$R_F \equiv \frac{1}{f\tau_f}. \tag{8.9}$$

Note that because f varies with latitude, both R_0 and R_F vary with time.

An expansion of (8.7) in terms of the *geostrophic* wind alone can be made by substituting $\mathbf{V}$ as given by (8.7) into the terms involving $\mathbf{V}$ on the right side of the same equation, resulting in

$$\mathbf{V} = V_g + R_0 \hat{k} \times \frac{d\mathbf{V}_g}{dt} + R_F \hat{k} \times \mathbf{V}_g - R_0^2 \frac{d^2\mathbf{V}}{dt^2}$$
$$- R_0 R_F \frac{d\mathbf{V}}{dt} - R_F^2 \mathbf{V} - R_0 \frac{d\mathbf{V}}{dt}\frac{dR_0}{dt}.$$

By repeating the procedure, this may be written

$$
\begin{aligned}
\mathbf{V} = \mathbf{V}_g + R_0 \hat{k} \times \frac{d\mathbf{V}_g}{dt} + R_F \hat{k} \times \mathbf{V}_g - R_0^2 \frac{d^2 \mathbf{V}_g}{dt^2} \\
- R_0 R_F \frac{d\mathbf{V}_g}{dt} - R_F^2 \mathbf{V}_g - R_0 \frac{d\mathbf{V}_g}{dt} \frac{dR_0}{dt} \\
+ O(R_0^3) + O(R_F^3),
\end{aligned}
\tag{8.10}
$$

assuming that $\frac{dR_0}{dt}$ is no larger than R_0.

If $R_0 < 1$ and $R_F < 1$, we might expect that the series (8.10) converges. The order zero approximation (8.10) is just geostrophic balance:

$$\mathbf{V} \simeq \mathbf{V}_g,$$

while the order 1 approximation is called the *geostrophic momentum approximation.* Writing the order 1 approximation to (8.10) in dimensional form results in

$$\frac{d\mathbf{V}_g}{dt} + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} \simeq 0, \tag{8.11}$$

where it must be remembered that $\mathbf{F}$ has been assumed to be *at most* order R_0.

The approximation (8.11) is called the geostrophic momentum approximation because it consists in replacing the inertia of the actual wind by that of the geostrophic wind. This approximation is one component of a system of approximate relations.

The second fundamental approximation to the momentum equations is to approximation advection by *geostrophic advection.* The full geostrophic momentum term may be expanded to

$$\frac{d\mathbf{V}_g}{dt} = \frac{\partial \mathbf{V}_g}{\partial t} + (\mathbf{V}_g + \mathbf{V}_a) \cdot \nabla \mathbf{V}_g + \omega \frac{\partial \mathbf{V}_g}{\partial p}, \tag{8.12}$$

where $\mathbf{V}_a$ is the ageostrophic part of the wind field, and

$$\omega \equiv \frac{dp}{dt} \tag{8.13}$$

is called the *pressure velocity* (or just "omega") and is proportional to the vertical component of velocity.

By (8.10), it is clear that

$$\frac{|\mathbf{V}_a|}{|\mathbf{V}_g|} \sim O(R_0);$$

moreover, we have already shown (by definition!) that $\left|\frac{d\mathbf{V}_g}{dt}\right|$ is $O(R_0)$ compared to $f\hat{k} \times \mathbf{V}$, so that to be consistent with the order of approximation, we need to drop the term $\mathbf{V}_a$ that appears in (8.13). In addition, the mass continuity equation in hydrostatic, pressure coordinates is

$$\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \tag{8.14}$$

which can also be written as

$$\nabla \cdot \mathbf{V}_g + \nabla \cdot \mathbf{V}_a + \frac{\partial \omega}{\partial p} = 0. \tag{8.15}$$

From the definition of geostrophic wind, (8.2),

$$\nabla \cdot \mathbf{V}_g = -\frac{\beta}{f^2}\frac{\partial \varphi}{\partial x} = -\frac{\beta}{f} v_g, \tag{8.16}$$

where v_g is the meridional component of the geostrophic wind, and

$$\beta \equiv \frac{df}{dy}. \tag{8.17}$$

Comparing (8.16) to (8.15), it will be seen that if

$$\frac{\beta L_y}{f} \lesssim 0(R_0), \tag{8.18}$$

then

$$\left| \frac{\omega}{\Delta p} \right| \lesssim R_0 \left| \frac{\mathbf{V}_g}{L} \right|, \tag{8.19}$$

where L_y is a typical meridional scale over which the flow varies, L is an overall horizontal scale of flow variation, and Δp is a pressure scale over which ω varies. If (8.19) is met, then we can also neglect the term involving ω in (8.12), which becomes

$$\frac{d\mathbf{V}_g}{dt} \simeq \frac{\partial \mathbf{V}_g}{\partial t} + \mathbf{V}_g \cdot \nabla \mathbf{V}_g.$$

Using this in (8.11) gives us the *quasi-geostrophic momentum equation:*

$$\frac{\partial \mathbf{V}_g}{\partial t} + \mathbf{V}_g \cdot \nabla \mathbf{V}_g + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} = 0. \tag{8.20}$$

The accuracy of (8.20) depends both on the smallness of R_0 and on condition (8.18).

The final element of this series of approximations is made to the thermodynamic equation, which may be written

$$\frac{\partial \ln \theta}{\partial t} + \mathbf{V} \cdot \nabla \ln \theta + \omega \frac{\partial \ln \theta}{\partial p} = \dot{Q}, \tag{8.21}$$

where for atmospheric applications, θ is the potential temperature and

$$\dot{Q} = \frac{\dot{H}}{c_p T},$$

where $\dot{H}$ is the heating and c_p is the heat capacity at constant pressure. For the ocean, θ is the potential density and Q is its source, divided by the potential density itself.

It may at first seem that the approximation to (8.21) that is consistent with the approximation we made to the momentum equation is to drop the ageostrophic advection and the term involving ω in (8.21), but this is not the case because in rotating stratified flows, the vertical gradient of θ scales very differently from its horizontal gradient. To see this, let's compare the magnitude of the horizontal and vertical advection terms in (8.21). The magnitude of the horizontal advection is approximately

$$|\mathbf{V} \cdot \nabla \ln \theta| \simeq \left| v_g \frac{\partial \ln \theta}{\partial y} \right| = \left| \frac{f}{g} v_g \frac{\partial u_g}{\partial z} \right|, \tag{8.22}$$

where we have used the thermal wind equation, and u_g is a typical geostrophic velocity scale. The magnitude of the vertical advection term is

$$\left| \omega \frac{\partial \ln \theta}{\partial p} \right| \sim \left| R_0 u_g \frac{N^2 h}{gL} \right|, \tag{8.23}$$

where we have used the hydrostatic relation, the scaling relation (8.19), h is a typical vertical scale of variation of the flow, and N is the *buoyancy (or Brünt-Väisälä) frequency,* defined

$$N^2 \equiv g \frac{\partial \ln \theta}{\partial z}. \tag{8.24}$$

Now the ratio of the magnitudes of the vertical and horizontal advection terms in the thermodynamic equation is

$$R \equiv R_0 \frac{N^2 h}{f \frac{\partial u_g}{\partial z} L}. \tag{8.25}$$

As we will see shortly, the *deformation radius* in quasi-geostrophic flows is

$$L_D = h\frac{N}{f},$$

so if L scales with L_D in (8.25),

$$R \simeq R_0 Ri^{1/2},$$

where Ri is the Richardson number,

$$Ri \equiv \frac{N^2}{\left(\frac{\partial u_g}{\partial z}\right)^2}.$$

In both the atmosphere and the ocean, Ri is an *order one* quantity, because the Richardson number is quite large. For this reason, we must retain the vertical advection term in (8.21), and for consistency, we expand $\ln\theta$ as

$$\ln\theta = \ln\overline{\theta}(p) + \ln\theta'(x, y, p, t), \tag{8.26}$$

with the scaling relation

$$\frac{\partial \ln\theta'}{\partial p} = O(R_0)\frac{\partial \ln\overline{\theta}}{\partial p}. \tag{8.27}$$

Then (8.21) is approximated by

$$\frac{\partial \ln\theta'}{\partial t} + \mathbf{V}_g \cdot \ln\theta' + \omega\frac{\partial \ln\overline{\theta}}{\partial p} = \dot{Q}. \tag{8.28}$$

(Note that Q is permitted to be order 1.)

Summary of quasi-geostrophic system:

The quasi-geostrophic equations may be summarized:

$$D_g \mathbf{V}_g + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) = \mathbf{F}, \tag{8.29}$$

$$D_g \theta + \omega \frac{d\overline{\theta}}{dp} = \dot{Q}, \tag{8.30}$$

$$\boldsymbol{\nabla} \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \tag{8.31}$$

$$\mathbf{V}_g = \frac{1}{f} \hat{k} \times \nabla \varphi, \tag{8.32}$$

$$\frac{\partial \varphi}{\partial p} = \begin{cases} -\frac{R}{p} \left(\frac{p}{p_0} \right)^{R/c_p} \theta & \text{atmosphere,} \\ -G\sigma & \text{ocean.} \end{cases} \tag{8.33}$$

In this set, the geostrophic operator is defined

$$D_g \equiv \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla},$$

and $\mathbf{F}$ is assumed to be of order R_0. In (8.33) G is a function of p that depends on the equation of state for sea water, and θ, σ, and φ (except where overbarred) are deviations from the basic state values of those quantities.

9. Quasi-geostrophic potential vorticity

The quasi-geostrophic system is at once more manageable and more intuitive if it is cast in the form of a potential vorticity conservation law and an invertibility principle. The potential vorticity conservation law can be obtained by combining a vorticity equation with the thermodynamic equation. The former can be obtained by taking the curl of (8.29), with the result

$$D_g \zeta_g + \beta v_g - f_0 \frac{\partial \omega}{\partial p} - \hat{k} \cdot \nabla \times \mathbf{F} = 0, \tag{9.1}$$

where

$$\zeta_g \equiv \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla^2 \varphi + O(R_0) \tag{9.2}$$

is the *geostrophic relative vorticity.* Note that each of the terms in (9.1) is now of $O(R_0)$, and that to be consistent with this, we have replaced the variable f with a mean value in (9.1) and we also need to drop the $O(R_0)$ term in (9.2). The β term in (9.1) should also be replaced by a mean value. Thus to $O(R_0)$, (9.1) becomes the *quasi-geostrophic vorticity equation:*

$$D_g \nabla^2 \varphi + \beta_0 \frac{\partial \varphi}{\partial x} - f_0^2 \frac{\partial \omega}{\partial p} - \hat{k} f_0 \nabla \times \mathbf{F} = 0. \tag{9.3}$$

The thermodynamic equation, (8.30), is first cast in a slightly different form by substituting (8.33) and dividing through by $d\overline{\theta}/dp$, with the result

$$D_g \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} + \omega = \frac{-\overline{\alpha} \dot{Q}}{\mathcal{S} \overline{\theta}} \tag{9.4}$$

with

$$\mathcal{S} \equiv \begin{cases} -\frac{\overline{\alpha}}{\overline{\theta}}\frac{d\overline{\theta}}{dp} & \text{atmosphere,} \\ -\frac{\overline{\alpha}}{\overline{\sigma}}\frac{d\overline{\sigma}}{dp} & \text{ocean,} \end{cases} \tag{9.5}$$

with $\overline{\alpha}$ given by the appropriate equation of state.

Note that (9.3) and (9.4) are functions of the variables φ and ω alone, given the distributions of Q and $\mathbf{F}$. We can form a predictive equation in the single variable φ by eliminating ω between the two equations. To do this, first take the derivative of (9.4) in p:

$$\frac{\partial}{\partial p} D_g \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} + \frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{\mathcal{S}\overline{\theta}}. \tag{9.6}$$

Now note that expanding the left side of (9.6) gives

$$\begin{aligned} \frac{\partial}{\partial p} D_g \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} &= D_g \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) + \frac{\partial \mathbf{V}_g}{\partial p} \cdot \nabla \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \\ &= D_g \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) + \frac{1}{\mathcal{S}} \frac{\partial \mathbf{V}_g}{\partial p} \cdot \nabla \frac{\partial \varphi}{\partial p}. \end{aligned} \tag{9.7}$$

(The $\mathcal{S}$ can be taken outside because it is a function of p alone.) But since

$$\nabla \varphi = -f \hat{k} \times \mathbf{V}_g,$$

(9.7) can be written

$$\begin{aligned} \frac{\partial}{\partial p} D_g \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} &= D_g \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) - \frac{f}{\mathcal{S}} \frac{\partial \mathbf{V}_g}{\partial p} \cdot \left(\hat{k} \times \frac{\partial \mathbf{V}_g}{\partial p} \right) \\ &= D_g \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right). \end{aligned}$$

Thus (9.6) can be re-expressed as

$$D_g \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) + \frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{\mathcal{S}\overline{\theta}}. \tag{9.8}$$

Multiplying this by f_0, dividing (9.3) by f_0, and adding the result gives

$$D_g \left[\frac{1}{f_0} \nabla^2 \varphi + \frac{\partial}{\partial p} \left(\frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) \right] + \beta_0 v_g = \hat{k} \cdot \nabla \times \mathbf{F} - f_0 \frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{\mathcal{S}}. \tag{9.9}$$

This can be written in a slightly different form by noting that $v_g = D_g y$:

$$\boxed{D_g q_p = \hat{k} \cdot \nabla \times \mathbf{F} - f_0 \frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{\mathcal{S}},} \tag{9.10}$$

where

$$\boxed{q_p \equiv \frac{1}{f_0} \nabla^2 \varphi + \beta_0 y + \frac{\partial}{\partial p} \left(\frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right)} \tag{9.11}$$

is the *pseudo potential vorticity.*

Note that, in contrast the Ertel's potential vorticity, q_p *is conserved following the geostrophic motion (or pressure surfaces).* Also note that *the invertibility relation (9.11) is a linear three-dimensional elliptic equation for φ.* Given certain boundary conditions, (9.10) and (9.11) constitute a closed system for φ, and the geostrophic wind and temperature (or density) perturbation can be recovered from (8.32) and (8.33).

Once again, there is a strong analogy with two-dimensional inviscid fluid dynamics, governed by

$$\frac{d\eta}{dt} = 0, \tag{9.12}$$

$$\eta = \nabla^2 \psi, \tag{9.13}$$

where ψ is the streamfunction of the two-dimensional flow. The difference lies in two places: Whereas (9.13) is exact, (9.11) relies on the quasi-geostrophic approximation, and contains a *three-dimensional* elliptic operator rather than a two-dimensional operator.

The inversion of elliptic operators like (9.11) or (9.13) encompasses the principle of *action at a distance:* A localized distribution of vorticity, or potential vorticity, yields a more global distribution of wind and temperature. Solutions of both (9.11) and (9.13), because they are linear operators, are *linearly superposable.* One useful technique for carrying out the inversion is using the method of Green's functions. For a point vortex in a two-dimensional flow, the solution of

$$\nabla^2\psi = A\delta(r),$$

where r is the radius from the source and A is the amplitude, is

$$\psi = -\frac{A}{2\pi}\ln r,$$

so that the tangential velocity, $\partial\psi/\partial r$, decays away from the point source as $1/r$. The action-at-a-distance principle is very analogous to the relationship between point changes and electric fields in electrostatics.

An analogous relation holds for the relationship between q_p and φ, as given by (9.11). To see this, let us first divide the q_p field according to

$$q_p = q_p' + \beta y,$$

so that, from (9.11),

$$q'_p = \frac{1}{f_0}\nabla^2\varphi + \frac{\partial}{\partial p}\left(\frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right). \tag{9.14}$$

In the special case that $\mathcal{S}$ is constant (equal to $\mathcal{S}_0$), (9.14) becomes

$$q'_p = \frac{1}{f_0}\nabla^2\varphi + \frac{f_0}{\mathcal{S}}\frac{\partial^2\varphi}{\partial p^2}. \tag{9.15}$$

Now suppose we scale the horizontal distances in the system by

$$x, y \to S_0^{1/2} f_0^{-1}\Delta p(x, y), \tag{9.16}$$

where Δp is some pressure scale, and scale pressure by Δp as well:

$$p \to (\Delta p)p. \tag{9.17}$$

Then (9.15), with the new independent variables, can be written

$$\frac{S_0\Delta p^2}{f_0}q'_p = \nabla_3^2\varphi, \tag{9.18}$$

where the notation ∇_3^2 is used to indicate the three-dimensional Laplacian operator. A three-dimensional point potential vortex of amplitude $Af_0/(S_0(\Delta p)^2)$ is associated with the geopotential distribution

$$\varphi = -\frac{A}{4\pi r}, \tag{9.19}$$

showing that the pressure distribution falls off inversely with radius from the source. The geostrophic velocities and temperatures fall off as $1/r^2$ in the horizontal and vertical directions, respectively, and form dipole fields oriented horizontally and vertically, respectively.

Note that the inversion of both (9.11) and (9.13) results in streamfunction anomalies of the opposite sign of the vorticity anomalies (or of the opposite sign of q'_p/f_0, in the quasi-geostrophic case).

We now have in hand the core elements of a mode of thinking about the dynamics of quasi-balanced flows: the twin principles of potential vorticity conservation and invertibility. In the simplest balance approximation, quasi-geostrophy, the quantity that is conserved (to order Rossby number) is the pseudo potential vorticity, given by (9.11), and this is a linear elliptic function of the perturbation, φ, of the geopotential from its basic state value. Pseudo potential vorticity is conserved, according to (9.10), *following the geostrophic flow on pressure surfaces* (as opposed to the actual, three-dimensional flow).

10. An example

Let us now turn to a specific example of the application of "potential vorticity" thinking, both as a mathematical physics framework and as a way of conceptualizing quasi-balanced dynamics. In this example, we shall linearize the conservation equation (9.10) about a state of constant mean flow and constant north-south gradient of pseudo potential vorticity, and look for strictly two-dimensional inviscid modal solutions of the linear equations.

This basic state is illustrated in Figure 10.1a. We divide the pseudo potential vorticity and flow fields into mean and perturbation parts, according to

$$q_p = \beta y + q_p', \tag{10.1}$$

$$\mathbf{V} = u_0 \hat{\imath} + \mathbf{V}' \tag{10.2}$$

and substitute these into the conservation equation (9.10), neglecting friction and heating. This results in the equation

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right) q_p' + v'\beta + \mathbf{V}' \cdot \nabla q_p' = 0. \tag{10.3}$$

We now seek solutions to (10.3) under the special circumstance that $|\mathbf{V}'| \ll |u_0|$ and $|q_p'| \ll \beta y$. In this case, the quadric term (the last term) in (10.3) may be neglected in comparison to the other terms, and (10.3) thus may be approximated by

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right) q_p' + \beta v' = 0. \tag{10.4}$$

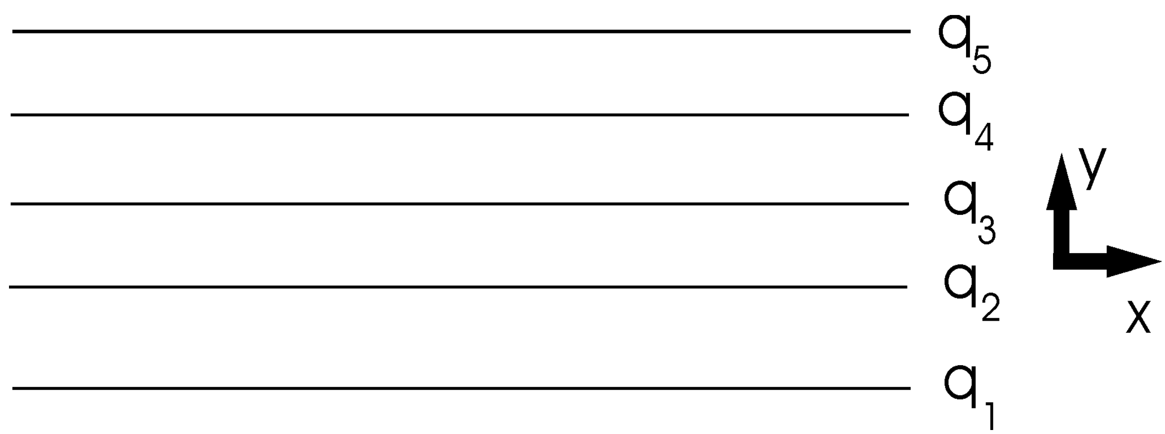

Figure 10.1a

Also, according to (10.1) and (9.11), q'_p is given by

$$q'_p = \frac{1}{f_0}\nabla^2\varphi + \frac{\partial}{\partial p}\left(\frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right), \tag{10.5}$$

while v' in (10.4) is given by the geostrophic relation

$$v' = \frac{1}{f_0}\frac{\partial\varphi}{\partial x}. \tag{10.6}$$

We now confine ourselves to strictly two-dimensional perturbations (so that I can sketch the fields on a piece of paper) of a *modal* character:

$$\varphi' = Ae^{ik(x-ct)+ily}, \tag{10.7}$$

where A is some undetermined amplitude, k and l are wavenumbers in the $\hat{\imath}$ and $\hat{\jmath}$ directions, respectively, and c is a phase speed in the $\hat{\imath}$ direction. Substitution of (10.7) into (10.5) and (10.6), and using (10.4) yields a *dispersion relation:*

$$c = u_0 - \frac{\beta}{k^2 + l^2}. \tag{10.8}$$

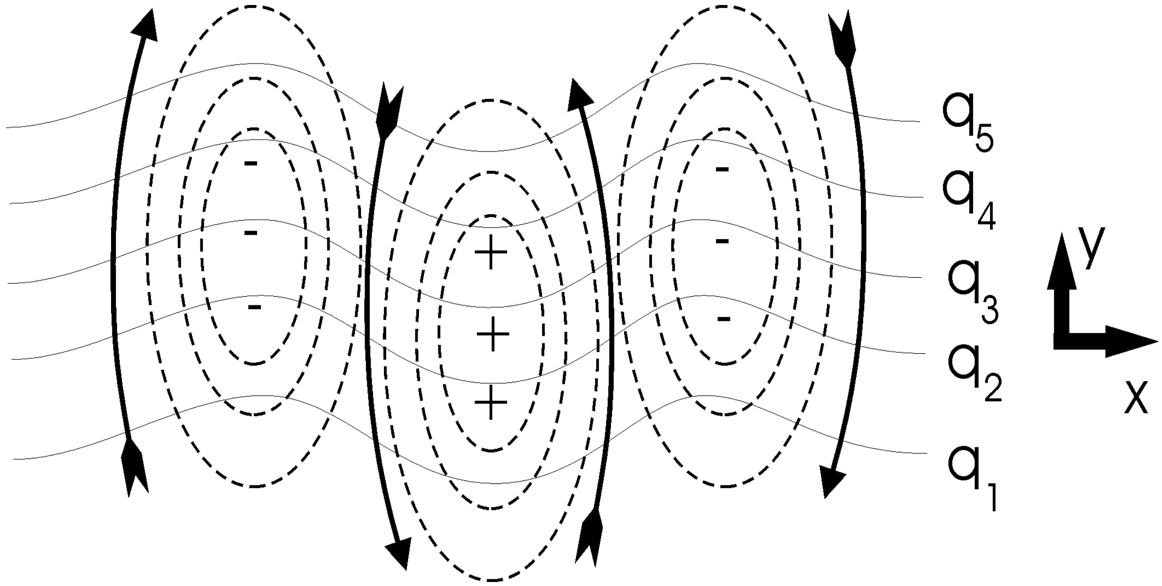

Figure 10.1b

These represent plane waves *travelling westward with respect to the background flow if β is positive.* These are called *Rossby waves.* Their dynamics can be visualized with the aid of Figure 10.1b. Northward perturbations of the q_p contours are associated with negative perturbations of q_p, while southward deflections produce positive q_p perturbations. By invertibility, these are associated with actual vorticity perturbations of the same sign, and geopotential (pressure) perturbations of the opposite sign. The geostrophic flow associated with these perturbations (see Figure 10.1b) acts to advect q_p in quadrature with the q_p perturbations, with the advection lagging the anomaly by 1/4 wave cycle. This shows that the wave must move westward relative to any background flow that may be present.

11. Boundary conditions

This last example is particularly simple and in fact represents the same solution that would have been obtained using the linearized form of the exact two-dimensional equations (9.12) and (9.13). By assuming infinitely periodic solutions, we did not have to worry about applying boundary conditions when we inverted (10.5). But in general, it will be necessary to apply boundary conditions. A particular problem arises in applying conditions at horizontal boundaries at the top and/or bottom of fluid systems. Here, the standard Neumann or Dirichlet conditions would amount to a specification of either the geopotential, φ, or $\partial\varphi/\partial p$, which is proportional to temperature by the hydrostatic equation. Specifying φ at the surface would eliminate one of the important signals we are actually interested in predicting: atmospheric surface pressure and the geopotential height of the sea surface. Moreover, specifying temperature also negates the prediction of an important quantity. Therefore, typically we choose to actually *solve a predictive equation for temperature at a horizontal boundary.* Assuming that $\omega = 0$ on such a boundary, the appropriate *dynamical boundary condition* is (9.4):

$$D_g \frac{\partial \varphi}{\partial p} = \frac{-\overline{\alpha}\dot{Q}}{\overline{\theta}}. \tag{11.1}$$

We shall see that much of the dynamics of atmospheric and oceanic quasi-balanced circulations enters through the dynamical boundary conditions (11.1).

But how should we think about the boundary conditions when inverting (10.5)? The Green's functions approach works well with strict Neumann or Dirichlet conditions because image points can be used. But dynamical boundary conditions like (11.1) usually give space- and time-varying boundary conditions. But a trick borrowed from electrostatics is again useful: We replace the actual boundary temperature perturbation with a zero value, but add point "charges" of potential vorticity just inside the boundary, in analogy to the concept of bound charge. To see how this works, let us integrate (10.5) from a lower boundary at pressure p_0 to a short distance above the boundary:

$$\int_{p_0-\epsilon}^{p_0} q_p' \, dp = \int_{p_0-\epsilon}^{p_0} \frac{1}{f_0}\nabla^2\varphi \, dp + \frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\bigg|_{p_0-\epsilon}^{p_0} . \tag{11.2}$$

Now in the limit of $\epsilon \to 0$, all the terms in (11.2) vanish because they are nonsingular. But suppose we *artificially* replace the actual temperature perturbation, $\partial\varphi/\partial p$, by zero at the boundary. Then (11.2) becomes

$$\int_{p_0-\epsilon}^{p_0} q_p' \, dp = -\frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\bigg|_{p_0-\epsilon} , \tag{11.3}$$

so that (11.2) can only be satisfied in this case if q_p' behaves like a delta function near the boundary.

From this development it follows that inverting (10.5) with an inhomogeneous boundary condition on temperature is equivalent to inverting it with a homogeneous boundary condition on temperature but inserting a delta function q_p' anomaly next to the boundary. From (11.3) and the hydrostatic equation (8.33), this delta function is

$$q'_p = \delta(p_0 - p) \times \begin{cases} \left.\frac{f_0}{\mathcal{S}_0}\frac{R}{p_0}\theta'\right|_{p_0} & \text{atmosphere} \\ \left.\frac{f_0}{\mathcal{S}_0}G_0\sigma'\right|_{p_0} & \text{ocean,} \end{cases} \tag{11.4}$$

where δ is the delta function. This equivalence is valid at a *lower* boundary. At an *upper* boundary, a similar development leads to

$$q'_p = \delta(p - p_t) \times \begin{cases} \left.-\frac{f_0}{\mathcal{S}_t}\frac{R}{p_t}\left(\frac{p_t}{p_0}\right)^{R/c_p}\theta'\right|_{p_t} & \text{atmosphere} \\ \left.-\frac{f_0}{\mathcal{S}_t}G_t\sigma'\right|_{p_t} & \text{ocean,} \end{cases} \tag{11.5}$$

where p_t is the pressure on the upper boundary. *Note the minus signs in (11.5).*

This device is of enormous *conceptual* significance, because it tells us that boundary temperature anomalies behave like delta function q_p anomalies just inside the boundary. In the case of a lower boundary, the temperature anomalies behave like delta function q_p anomalies of the *same* sign, whereas at upper boundaries they behave like q_p anomalies of the *opposite* sign.

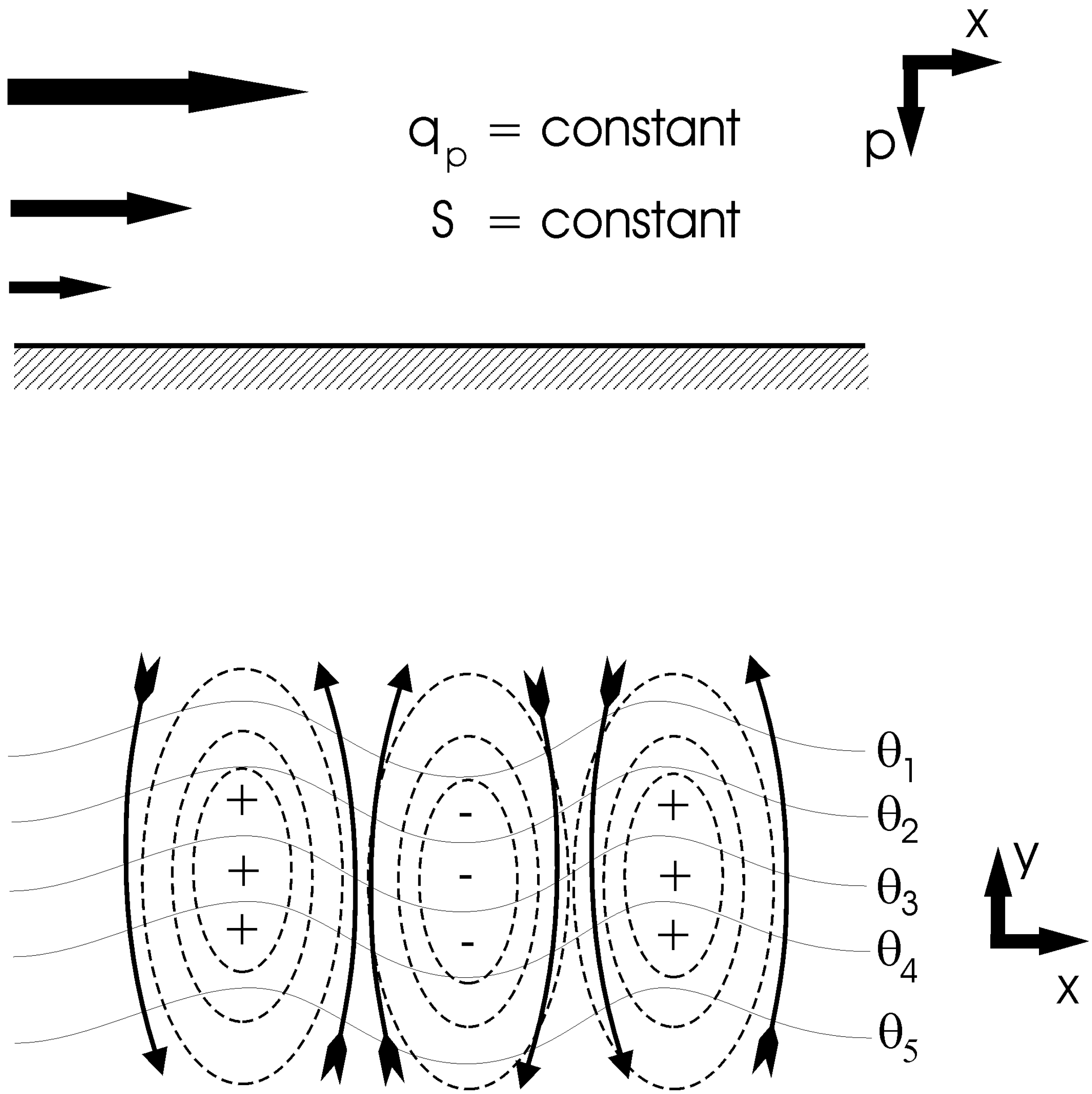

Figure 12.1

12. Eady edge waves

A specific example illustrating the importance of dynamical boundary conditions is a subspecies of Rossby wave, called the *Eady edge wave.* Suppose we have a flow of a quasi-geostrophic fluid in a semi-infinite domain, as illustrated in Figure 12.1a.

This fluid, we shall suppose, has constant pseudo potential vorticity and static stability, $\mathcal{S}$, and no sources or sinks of q_p, so that

$$q_p' = 0$$

everywhere, for all time. Then (10.5) becomes

$$\frac{1}{f_0}\nabla^2\varphi + \frac{f_0}{\mathcal{S}}\frac{\partial^2\varphi}{\partial p^2} = 0. \tag{12.1}$$

Now suppose that at the lower boundary, the temperature consists of a (negative, shall we say) north-south temperature gradient plus superimposed perturbations:

$$\theta = \overline{\theta}_y y + \theta', \tag{12.2}$$

where $\overline{\theta}_y$ is the mean temperature gradient and that the wind consists of a zonal flow plus perturbations to it:

$$\mathbf{V} = u_0\hat{\imath} + \mathbf{V}'. \tag{12.3}$$

The dynamical boundary condition, (11.1), becomes, after assuming that the perturbations are small and neglecting quadratic terms in the perturbation variables,

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\frac{\partial\varphi}{\partial p} + \gamma\frac{\partial\varphi}{\partial x} = 0 \quad \text{at} \quad p = p_0, \tag{12.4}$$

where we have used the geostrophic wind relation (10.6), and

$$\gamma \equiv -\overline{\theta}_y R/f_0 p_0, \tag{12.5}$$

using (8.33). (Had we used an example from the ocean, γ would have been $-G\overline{\sigma}_y$ evaluated at a suitable pressure level.)

For simplicity, we shall assume that the temperature perturbations vanish at the top of the fluid, so that

$$\frac{\partial \varphi}{\partial p} = 0 \quad \text{at} \quad p = 0. \tag{12.6}$$

Since the fluid is semi-infinite, we are entitled to look for model solutions that are periodic in x, y, and time, so let

$$\varphi = \hat{\varphi}(p) e^{ik(x-ct)+ily}, \tag{12.7}$$

where k, l, and c have the same meanings as before. Substituting (12.7) into (12.1), (12.4) and (12.6) gives

$$\frac{d^2\hat{\varphi}}{dp^2} - \frac{\mathcal{S}}{f_0^2}(k^2 + l^2)\hat{\varphi} = 0, \tag{12.8}$$

subject to the boundary conditions

$$(u_0 - c)\frac{d\hat{\varphi}}{dp} + \gamma\hat{\varphi} = 0 \quad \text{on} \quad p = p_0, \tag{12.9}$$

$$\frac{d\hat{\varphi}}{dp} = 0 \quad \text{on} \quad p = p_t. \tag{12.10}$$

The system comprised of (12.8)–(12.10) is a closed eigenvalue problem with the solution

$$\hat{\varphi} = A\cosh(rp), \tag{12.11}$$

where A is an arbitrary amplitude, with

$$r^2 \equiv \frac{\mathcal{S}}{f_0}(k^2 + l^2). \tag{12.12}$$

This can only satisfy the lower boundary condition (12.9) if

$$c = u_0 + \frac{\gamma}{r}\,\mathrm{ctnh}(rp_0), \tag{12.13}$$

with r given by (12.12). Provided γ is positive (which, form (12.5), implies a negative temperature gradient), the waves will travel eastward relative to the background flow. Note that, like Rossby waves, the relative phase speed increases without bound as the horizontal wavelength becomes large. It is also clear from (12.11) that the wave amplitude decreases more or less exponentially upward (i.e., with decreasing pressure). The wave is three-dimensional, but trapped at the lower boundary. Note that all the time dependence is in the lower boundary condition, (12.9).

The wave dynamics are conceptualized in Figure 12.1b. Where the isotherms are deflected northward, the temperature perturbation is positive, and vice versa; invertibility sees those temperature perturbations as potential vorticity perturbations of the same sign, so that positive vorticity is associated with positive θ perturbations. The associated geostrophic flow, illustrated in Figure 12.1b, is in quadrature with θ in such a way that the wave propagates eastward relative to any background flow. Remember that there are no q_p perturbations in the interior of the fluid in the example, and that the boundary temperature anomalies act like delta function q_p anomalies, so the wave is evanescent with height away from the boundary. That is why the wave is called an *edge wave.*

13. The superposition principle

In the quasi-geostrophic system, the relationship between q_p anomalies and φ anomalies is linear, so that distributions of φ associated with individual anomalies of q_p can be superposed to form the full φ field associated with the full q_p distribution. The hydrostatic and geostrophic relations are also linear, so that perturbations of velocity and temperature also just superpose linearly. But *energy,* on the other hand, is a quadratic, and thus does not superpose linearly. This has important implications for energy transformations in quasi-balanced flows in general, and quasi-geostrophic flow, in particular. To see some of these implications, first form an energy integral for quasi-geostrophic flows. The equation for kinetic energy can be obtained by taking the dot product of the geostrophic flow vector, $\mathbf{V}_g$, with the quasi-geostrophic momentum equation (8.29) with the result

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) \frac{1}{2}|\mathbf{V}_g|^2 = -f\mathbf{V}_g \cdot \hat{k} \times \mathbf{V} + \mathbf{V}_g \cdot \mathbf{F}, \tag{13.1}$$

where we have made use of the vector identity

$$\mathbf{A} \cdot \hat{k} \times \mathbf{A} = 0,$$

for any vector $\mathbf{A}$. By using the geostrophic relation (8.32) in (13.1), the latter may be written

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) \frac{1}{2}|\mathbf{V}_g|^2 &= -\mathbf{V} \cdot \nabla\varphi + \mathbf{F} \cdot \mathbf{V}_g \\ &= -\nabla \cdot (\mathbf{V}\varphi) - \varphi\frac{\partial \omega}{\partial p} + \mathbf{F} \cdot \mathbf{V}_g, \end{aligned} \tag{13.2}$$

where we have made use of the mass continuity equation, (8.31). This is the quasi-geostrophic kinetic energy equation.

An equation for potential and internal energy can be formed by multiplying (9.4) by $\partial\varphi/\partial p$:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) \frac{1}{2\mathcal{S}} \left(\frac{\partial \varphi}{\partial p}\right)^2 + \omega \frac{\partial \varphi}{\partial p} = -\frac{\partial \varphi}{\partial p} \overline{\alpha} \frac{\dot{Q}}{\mathcal{S}\overline{\theta}}. \tag{13.3}$$

An equation for total energy associated with perturbations can be formed by summing (13.2) and (13.3):

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) \left[\frac{1}{2}|\mathbf{V}_g|^2 + \frac{1}{2}\frac{1}{\mathcal{S}} \left(\frac{\partial \varphi}{\partial p}\right)^2\right] = -\nabla \cdot (\mathbf{V}\varphi) - \frac{\partial}{\partial p}(\omega\varphi) + \mathbf{F} \cdot \mathbf{V}_g - \frac{\partial \varphi}{\partial p} \overline{\alpha} \frac{\dot{Q}}{\mathcal{S}\overline{\theta}}. \tag{13.4}$$

Energy is locally changed by a divergence of the flux (by the actual wind) of perturbation geopotential, by heating and by friction.

We next integrate (13.4) over a three-dimensional volume defined in such a way that there is no net flux of geopotential or energy itself across its boundaries. This integration (after applying the divergence theorem) then results in

$$\int_v \frac{\partial E_p}{\partial t} dV = \int_v \left(\mathbf{F} \cdot \mathbf{V}_g - \frac{\partial \varphi}{\partial p} \overline{\alpha} \frac{\dot{Q}}{\mathcal{S}\overline{\theta}}\right), \tag{13.5}$$

where E_p is the pseudo energy, defined

$$E_p \equiv \frac{1}{2}|\mathbf{V}_g|^2 + \frac{1}{2\mathcal{S}} \left(\frac{\partial \varphi}{\partial p}\right)^2. \tag{13.6}$$

This shows that the integral of E_p over a suitably defined volume is conserved in the absence of heating and friction.

We can now show that E_p is related to an integral of the pseudo potential vorticity. Begin with the perturbation pseudo potential vorticity defined by (9.14), multiply it by $-\varphi$ and integrate the result over a control volume on whose lateral sides either φ or its normal gradient vanishes:

$$-\int_v \varphi q_p' dV = -\frac{1}{f_0}\int_v \varphi \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) dV - f_0 \int_v \varphi \frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right) dV. \tag{13.7}$$

We next integrate the two terms on the right side of (13.7) by parts and use the geostrophic relations to get

$$\begin{aligned} -\int_v \varphi q_p' dV = {} & f_0 \int_v \left(|\mathbf{V}_g|^2 + \frac{1}{\mathcal{S}}\left(\frac{\partial\varphi}{\partial p}\right)^2\right) dV \\ & - \frac{1}{f_0}\int_v \left[\frac{\partial}{\partial x}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right] dV - f_0 \int_A \frac{1}{\mathcal{S}}\varphi\frac{\partial\varphi}{\partial p}\bigg|_{p_t}^{p_0} dA, \end{aligned} \tag{13.8}$$

where the last integral is over the horizontal areas bounding the top and bottom of the control volume.

By our assumption that either φ or its normal gradient vanishes at the lateral boundaries of the volume, the second term on the right side of (13.8) vanishes, and combining the last term on the right with the left side of (13.8) gives

$$-\int_v \varphi\left[q_p' - \frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\delta(p_0 - p) + \frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\delta(p - p_t)\right] = 2f_0 \int_v E_p\, dV, \tag{13.9}$$

where we have substituted (13.6).

Note that using the arguments presented in section 11, the left side of (13.9) is simply the integral of the product of φ with the pseudo potential vorticity, *including* the effective delta functions at horizontal boundaries that result when there are temperature perturbations there. This integral is proportional to the pseudo energy of the system.

A general conclusion that can be reached with the aid of (13.9) is that *bringing together like-signed potential vorticity anomalies (or their equivalent delta functions in the form of boundary θ' anomalies) entails an increase in the energy associated with the anomalies.* An example will suffice to show why this follows. Suppose we have two delta function q_p anomalies in an infinite domain, separated by a great distance, as in the top of Figure 13.1. Let us suppose that each delta function has an amplitude of 1 in some suitably normalized three-dimensional coordinates.

Now the inversion of the elliptic relationship between q_p' and φ will result in a field of φ that decays away from each of the two point potential vortices. Let us suppose that φ has been normalized in such a way that its value at the location of the point potential vortex is -1. Let us also suppose that the two vortices are so far apart that, for all practical purposes, the amplitude of the part of φ associated with one point potential vortex is zero at the location of the other point potential vortex. In that case, the integral at the left of (13.9) will have the value of 4π associated with each vortex, or 8π total.

Suppose that these two point potential vortices are brought together by some process and combined into a single point potential vortex, as at the bottom of Figure 13.1. Now there are 2 units of q_p' in the combined vortex, and the amplitude of φ

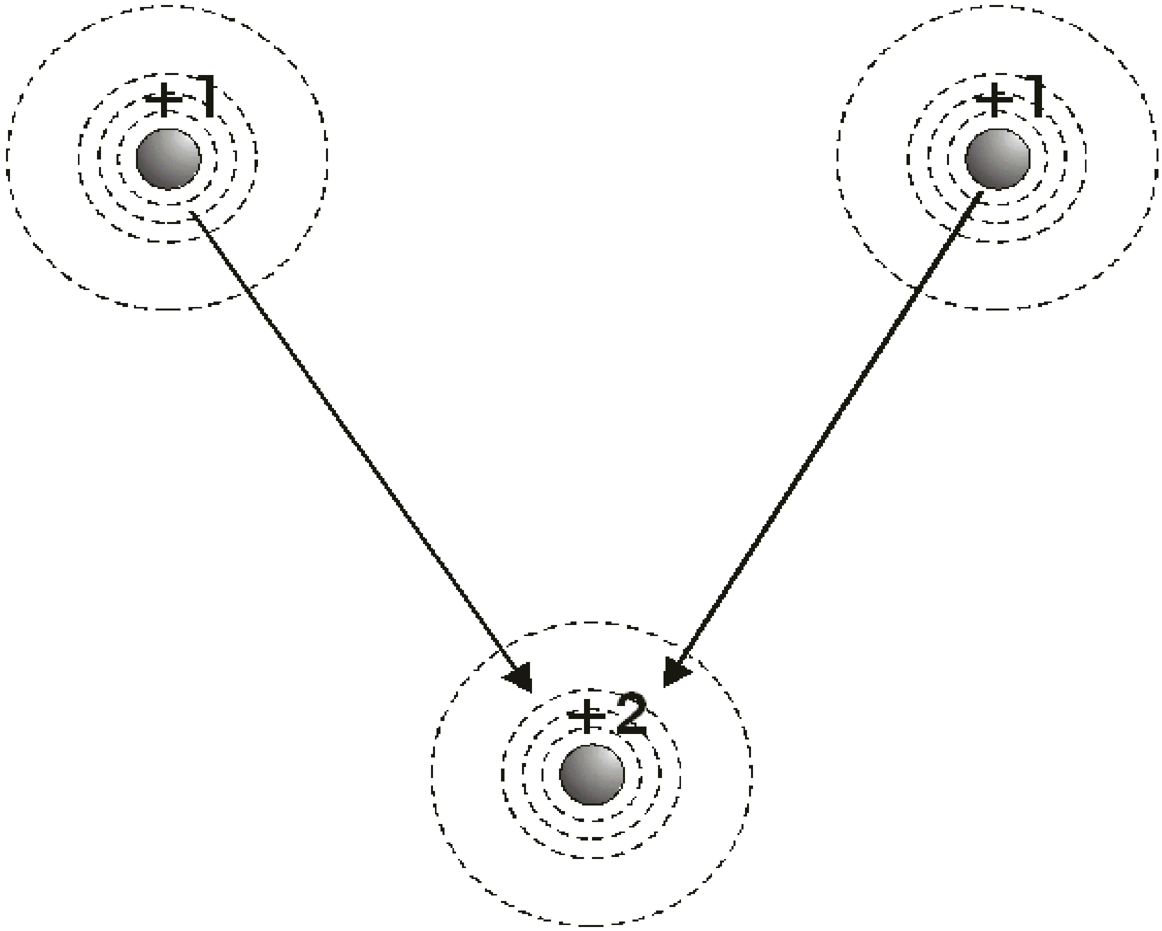

Figure 13.1

at the location of the combined vortex is thus -2. This gives 16π energy units, according to (13.9)—double what was there before!

Thus it takes a source of energy (from, say, the background flow) to rearrange potential vorticity into more compact masses, for which the energy anomaly is greater. For a given mass of potential vorticity, the maximum energy is achieved when the potential vorticity is concentrated in a sphere, in a coordinate system scaled by the deformation radius. The minimum energy content occurs when the potential vorticity is distributed in an infinitely long thread of zero thickness.

14. The secondary circulation

The quasi-geostrophic system in which pseudo potential vorticity is advected by the geostrophic flow and inverted to obtain the geostrophic flow, geopotential, and temperature perturbation, does not require for its solution explicit knowledge of the secondary circulation, including ω and the ageostrophic flow; but the existence of changing temperature and vorticity fields *implies* the existence of a secondary circulation.

To see this, suppose we have solved the advection and inversion equations for q_p at two different times separated by a small time increment, Δt. Then the quasi-geostrophic vorticity equation (9.3) directly implies that

$$f_0 \frac{\partial \omega}{\partial p} = \mathbf{V}_g \cdot \boldsymbol{\nabla}(\zeta_g + \beta y) + \frac{\zeta_g|_t^{t+\Delta t}}{\Delta t} - \hat{k} \cdot \nabla \times \mathbf{F}. \tag{14.1}$$

This means that *wherever the evolution of the q_p field implies a change of absolute vorticity following the flow, stretching (and/or friction) is implied.*

As an example, suppose that a weak positive q_p anomaly is embedded in a shear flow, as shown in Figure 14.1, which is deliberately placed in a coordinate system moving with the q_p anomaly.

Following a sample of air as it moves with the geostrophic wind (and assuming that the q_p anomaly is not so strong that it appreciably deflects the background geostrophic trajectories), the absolute vorticity first increases and then decreases,

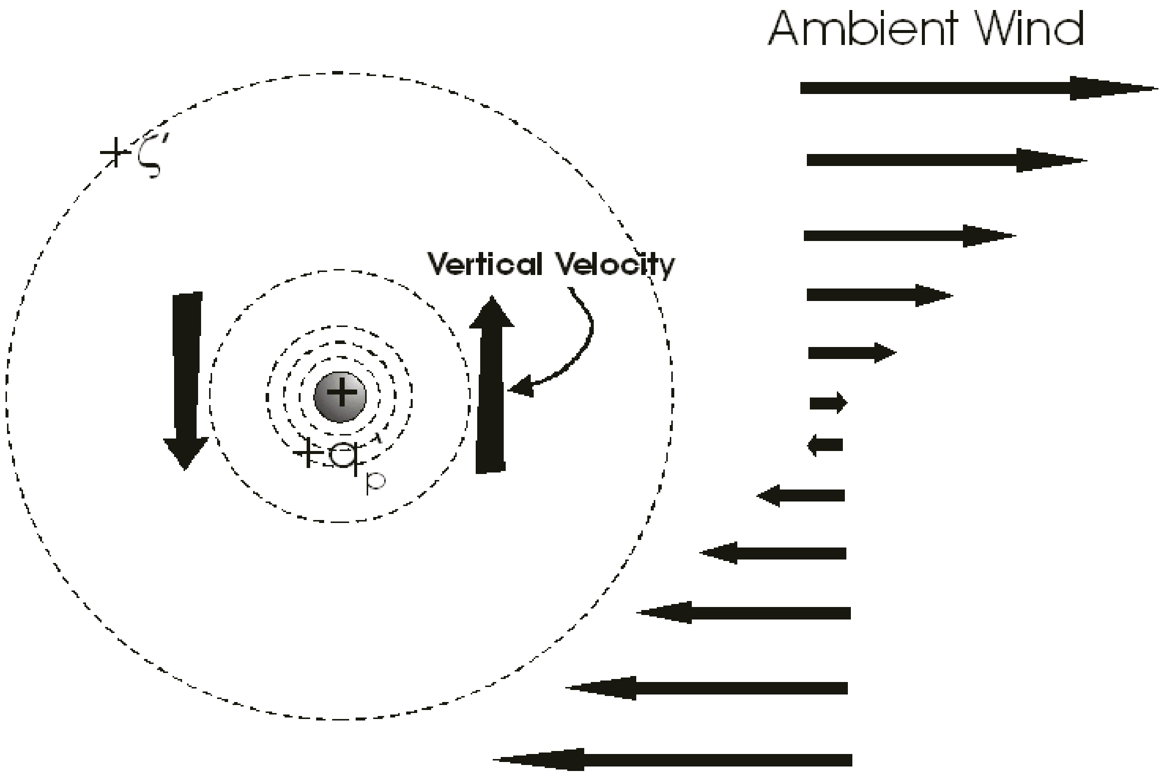

Figure 14.1

implying stretching and then shrinking of vertical columns. From this, the distribution of ω can be inferred.

Alternatively, one can solve a diagnostic equation for ω by eliminating the local time tendency terms from the quasi-geostrophic vorticity and thermodynamic equations. To do this, operate on (9.4) by $\mathcal{S}$ times the ∇^2 operator, and on (9.3) by $\partial/\partial p$ and subtract one from the other. The result may be written

$$\left(f_0^2\frac{\partial^2}{\partial p^2}+\mathcal{S}\nabla^2\right)\omega = f_0\frac{\partial}{\partial p}\left[\mathbf{V}_g\cdot\boldsymbol{\nabla}\eta_g\right]-\nabla^2\left[\mathbf{V}_g\cdot\boldsymbol{\nabla}\frac{\partial\varphi}{\partial p}\right]-\frac{\overline{\alpha}}{\overline{\overline{\theta}}}\nabla^2\dot{Q}-f_0\frac{\partial}{\partial p}(\hat{k}\cdot\boldsymbol{\nabla}\times\mathbf{F}). \tag{14.2}$$

This is the “classical” form of the *quasi-geostrophic ω equation.* Vertical velocity is associated with vertically varying geostrophic advection of vorticity and horizontal Laplacians of the geostrophic temperature advection, as well as with friction and heating. The non-Galilean invariant parts of the right side of (14.1) cancel, so that this is perhaps not the most useful form of the equation. A somewhat improved form may be obtained by noting from (9.11) that

$$\eta_g = q_p - f_0 \frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial \varphi}{\partial p}\right). \tag{14.3}$$

Substitution of this into (14.1) results, after some rearrangement of terms, in

$$\begin{aligned}\left(f_0^2 \frac{\partial^2}{\partial p^2} + \mathcal{S}\nabla^2\right)&\left(\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \boldsymbol{\nabla}\theta'\right) \\ &= f_0 \frac{\partial}{\partial p}\left(\mathbf{V}_g \cdot \boldsymbol{\nabla} q_p\right) - \frac{\overline{\alpha}}{\overline{\overline{\theta}}}\nabla^2 \dot{Q} - f_0 \frac{\partial}{\partial p}\left[\hat{k} \cdot \boldsymbol{\nabla} \times \mathbf{F}\right].\end{aligned} \tag{14.4}$$

The adiabatic quantity on the right side of (14.4) is just the variation with altitude of the geostrophic advection of pseudo potential vorticity, while the elliptic operator on the left now acts on the sum of ω and the geostrophic temperature advection. Unfortunately, the boundary conditions in inverting the elliptic operator are now inhomogeneous, since $\mathbf{V}_g \cdot \nabla\theta'$ may be nonzero on the boundaries. This can be "fixed" by the method of "bound charge," as before, so that (14.3) may be rewritten

$$\begin{aligned}\left(f_0^2 \frac{\partial^2}{\partial p^2} + \mathcal{S}\nabla^2\right)&\left(\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \boldsymbol{\nabla}\theta'\right) \\ &= f_0 \frac{\partial}{\partial p}\left[\mathbf{V}_g \cdot \boldsymbol{\nabla}\left(q_p + \delta(p_0 - p)\frac{f_0}{\mathcal{S}}\theta' - \delta(p - p_t)\frac{f_0}{\mathcal{S}}\theta'\right)\right] \\ &\quad - \alpha^2\nabla^2 Q - f_0 \frac{\partial}{\partial p}\left[\hat{k} \cdot \boldsymbol{\nabla} \times \mathbf{F}\right],\end{aligned} \tag{14.5}$$

where it is understood that $\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \nabla\theta'$ is forced to be zero on horizontal boundaries when inverting (14.5). Once again we see that the system is forced by changes in q_p, and θ' at the boundaries.

15. Higher-order balance systems

The quasi-geostrophic system is adequate for very low Rossby number and Froude number flows, such as characterize the ocean even at the "mesoscale" (O(100 km)). The low Rossby-Froude number approximation in the atmosphere is not as good, and the quasi-geostrophic equations do not work as well, *quantitatively.* Higher-order balance approximations are needed for an accurate diagnosis of quasi-balanced dynamical processes in the atmosphere, or for numerical weather prediction, if we were to use a quasi-balanced system. (For various reasons, balanced equations have been abandoned as a basis for NWP and the full set of "primitive," or hydrostatic, equations are used.)

One basis of higher-order balance systems involves approximating the *horizontal divergence as small compared to the vertical component of vorticity* in a fluid flow. This approximation also leads to a system in which the velocity field is instantaneously related to the mass distribution.

Begin with the hydrostatic approximation to the momentum equations in pressure coordinates:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \omega \frac{\partial \mathbf{V}}{\partial p} = -\nabla \varphi - f \hat{k} \times \mathbf{V} + \mathbf{F}. \tag{15.1}$$

Now operate on the above with the horizontal divergence operator, $\boldsymbol{\nabla}_2$. The result may be written

$$\frac{dD}{dt} = -\nabla^2 \varphi + (f + \zeta)\zeta - \beta u - \frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p} - S^2 + \boldsymbol{\nabla}_2 \cdot \mathbf{F}, \tag{15.2}$$

where D is the horizontal divergence, $\boldsymbol{\nabla}_2 \cdot \mathbf{V}$, ζ is the vertical component of vorticity, and

$$S^2 \equiv \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \qquad (15.3)$$
$$\beta \equiv \frac{df}{dy}.$$

If we assume that

$$\left|\frac{dD}{dt}\right| \ll |(f+\zeta)\zeta|\,, \qquad (15.4)$$

then (15.2 may be approximated by the *nonlinear balance equation:*

$$\nabla^2\varphi = (f+\zeta)\zeta - \beta u - S^2 - \frac{\partial\omega}{\partial x}\frac{\partial u}{\partial p} - \frac{\partial\omega}{\partial y}\frac{\partial v}{\partial p} + \boldsymbol{\nabla}_2 \cdot \mathbf{F}. \qquad (15.5)$$

Unlike the geostrophic relations, (15.5) represents the geopotential field as a *nonlinear* function of the velocity distribution. Note that we have assumed that the total derivative of D is small compared to *all* the terms in (15.5).

At the same time, the hydrostatic approximation to Ertel's potential vorticity is

$$q = -g\left[(f+\zeta)\frac{\partial\theta}{\partial p} - \frac{\partial v}{\partial p}\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial p}\frac{\partial\theta}{\partial y}\right], \qquad (15.6)$$

and conservation of potential vorticity is governed by (cf. 5.10)

$$\frac{\partial q}{\partial t} = -\mathbf{V}\cdot\boldsymbol{\nabla} q - \omega\frac{\partial q}{\partial p} + \alpha(\nabla\times\mathbf{F})\cdot\boldsymbol{\nabla}\theta + \alpha(2\boldsymbol{\Omega} + \boldsymbol{\nabla}\times\mathbf{V})\cdot\boldsymbol{\nabla}\frac{d\theta}{dt}. \qquad (15.7)$$

Also, remember that θ is related hydrostatically to φ via (8.33), whose atmosphere part we write

$$\pi\frac{\partial\varphi}{\partial p} = -\theta, \qquad (15.8)$$

where

$$\pi \equiv p\left(\frac{p_0}{p}\right)^{R/c_p}/R. \tag{15.9}$$

To this we add the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0. \tag{15.10}$$

Clearly, the five equations (15.5)–(15.8) and (15.10) contain the six dependent variables u, v, ω, θ, φ, and q and so do not constitute a closed system. Progress can be made by ordering the flow according to the relative magnitudes of its irrotational and nondivergent components. We first express the horizontal flow components in terms of a mass streamfunction, ψ, and a velocity potential, χ:

$$\begin{aligned} u &= -\frac{\partial \psi}{\partial y}+\frac{\partial \chi}{\partial x}, \\ v &= \frac{\partial \psi}{\partial x}+\frac{\partial \chi}{\partial y}, \end{aligned} \tag{15.11}$$

where, by (15.10),

$$\nabla^2\chi = -\frac{\partial \omega}{\partial p}. \tag{15.12}$$

Using (15.11) and (15.8), the set (15.5)–(15.7) can be written

$$\begin{aligned} \nabla^2\varphi = (f+\nabla^2\psi)\nabla^2\psi + \beta\left(\frac{\partial \psi}{\partial y}-\frac{\partial \chi}{\partial x}\right) - S^2(\psi,\chi) \\ -\frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p}+\boldsymbol{\nabla}_2\cdot\mathbf{F}, \end{aligned} \tag{15.13}$$

$$\begin{aligned} q = g\Bigg[(f+\nabla^2\psi)\frac{\partial}{\partial p}\left(\pi\frac{\partial \varphi}{\partial p}\right)-\pi\frac{\partial^2\varphi}{\partial p\partial x}\frac{\partial}{\partial p}\left(\frac{\partial \psi}{\partial x}+\frac{\partial \chi}{\partial y}\right) \\ +\pi\frac{\partial^2\varphi}{\partial y\partial p}\frac{\partial}{\partial p}\left(\frac{\partial \chi}{\partial x}-\frac{\partial \psi}{\partial y}\right)\Bigg], \end{aligned} \tag{15.14}$$

$$\frac{\partial q}{\partial t} = \left(\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial x}\right)\frac{\partial q}{\partial x} - \left(\frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y}\right)\frac{\partial q}{\partial y} - \omega\frac{\partial q}{\partial p} \tag{15.15}$$
$$+ \text{ friction and heating.}$$

Consistent with the approximation given by (15.4), we assume that

$$|\chi| < |\psi|\,. \tag{15.16}$$

Consider an iterative process in which, in the first step, we take χ (and therefore ω) to be zero in (15.13)–(15.15). In that case, we have a closed system in the variables q, φ, and ψ, and, given appropriate boundary conditions, *we can step (15.15) forward one time step, then invert the system (15.13)–(15.14) to get ψ and φ at the next time step.* This gives us all the fields q, ψ, and φ at the beginning *and* end of a time step. Now, we introduce another equation, the thermodynamic equation, which can be written (backwards) in terms of φ:

$$\begin{aligned}\omega\frac{\partial}{\partial p}\left(\pi\frac{\partial \varphi}{\partial p}\right) = -\pi\left(-\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}\right)\frac{\partial^2 \varphi}{\partial p \partial x} - \pi\left(\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}\right)\frac{\partial^2 \varphi}{\partial p \partial y} \\ \frac{-\pi}{\Delta t}\frac{\partial \varphi}{\partial p}\Bigg|_t^{t+\Delta t} - Q,\end{aligned} \tag{15.17}$$

where Δt is the time step. Again taking χ to be zero on the right side of (15.17) we can now solve (15.17) for ω since we know φ and ψ at the beginning and end of the time step. We can then solve (15.12) for an estimate of χ, and start the whole time step over again, this time with a nonzero estimate for χ. This gives us new estimates of q, ψ, and φ at the end of the time step and therefore, from (15.17), new estimate of ω and χ. Although no formal proof has been developed that shows that the iteration converges, it seems to in practice. Experience shows that this method of solution of the equations describes atmospheric motions very well.

The ability of a system like this one to describe quite accurately atmospheric and oceanic flows raises an important philosophical question: Can general flows be separated into quasi-balanced parts, uniquely associated with the potential vorticity distribution, and everything that is left over after the balanced part is accounted for (e.g., inertia-gravity waves, three-dimensional turbulence)? The precise answer seems to be "no"; there will always be some interaction between the quasi-balanced and unbalanced flow components that render their separation imprecise. Nonetheless, these interactions are *usually* (but not always) weak in geophysical fluid flows, and so it is useful to talk about these components separately. Our *working* definition of quasi-balanced flow is *flow uniquely associated with and calculable from the potential vorticity distribution.*

16. Rossby waves

We have seen that the existence of potential vorticity gradients supports the propagation of a special class of waves known as *Rossby waves.* These waves are the principal means by which information is transmitted through quasi-balanced flows and it is therefore fitting to examine their properties in greater depth. We begin by looking at the classical problem of barotropic Rossby wave propagation on a sphere and continue with quasi-geostrophic Rossby waves in three dimensions.

a. Barotropic Rossby waves on a sphere

The vorticity equation for barotropic disturbances to fluid at rest on a rotating sphere is

$$\frac{d\eta}{dt} = 0, \tag{16.1}$$

where

$$\eta \equiv 2\Omega \sin\varphi + \zeta.$$

Here ζ is the relative vorticity in the z direction. Now the equation of mass continuity for two-dimensional motion on a sphere may be written

$$\frac{1}{a}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi}(v\cos\varphi)\right] = 0, \tag{16.2}$$

where u and v are the eastward and northward velocity components, λ and φ are longitude and latitude, and a is the (mean) radius of the earth. Using (16.2) we may define a velocity streamfunction ψ such that

$$u = -\frac{1}{a}\frac{\partial \psi}{\partial \varphi},$$

and (16.3)

$$v = \frac{1}{a\cos\varphi}\frac{\partial \psi}{\partial \lambda}.$$

The Eulerian expansion of (16.1) can be written

$$\frac{\partial \eta}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial \eta}{\partial \lambda} + \frac{v}{a}\frac{\partial \eta}{\partial \varphi} = 0,$$

or using (16.3),

$$\frac{\partial \eta}{\partial t} + \frac{1}{a^2\cos\varphi}\left[\frac{\partial \psi}{\partial \lambda}\frac{\partial \eta}{\partial \varphi} - \frac{\partial \psi}{\partial \varphi}\frac{\partial \eta}{\partial \lambda}\right] = 0. \tag{16.4}$$

We next linearize (16.4) about the resting state ($u = v = 0$), for which $\overline{\eta} = 2\Omega \sin\varphi$, giving

$$\frac{\partial \eta'}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi'}{\partial \lambda} = 0, \tag{16.5}$$

where the primes denote departures from the basic state.

In spherical coordinates,

$$\begin{aligned} \eta' = \zeta' &= \hat{k} \cdot \nabla \times \mathbf{V}' \\ &= \frac{1}{a^2 \cos^2\varphi}\left[\frac{\partial^2 \psi'}{\partial \lambda^2} + \cos\varphi \frac{\partial}{\partial \varphi}\left(\cos\varphi \frac{\partial \psi'}{\partial \varphi}\right)\right]. \end{aligned} \tag{16.6}$$

Let's look for modal solutions of the form

$$\psi' = \Psi(\varphi) e^{im(\lambda - \sigma t)},$$

where m is the zonal wavenumber and σ is an *angular* phase speed. Using this and (16.6) in (16.5) gives

$$\frac{d^2\Psi}{d\varphi^2} - \tan\varphi \frac{d\Psi}{d\varphi} - \left[\frac{2\Omega}{\sigma} + \frac{m^2}{\cos^2\varphi}\right]\Psi = 0. \tag{16.7}$$

This can be transformed into canonical form by transforming the independent variable using

$$\mu \equiv \sin\varphi,$$

yielding

$$(1-\mu^2)\frac{d^2\Psi}{d\mu^2} - 2\mu \frac{d\Psi}{d\mu} - \left[\frac{2\Omega}{\sigma} + \frac{m^2}{1-\mu^2}\right]\Psi = 0. \tag{16.8}$$

The only solutions of (16.8) that are bounded at the poles ($\mu = \pm 1$) have the form

$$\Psi = AP_m^n, \tag{16.9}$$

Table 16.1. Meridional Structure of $P_m^n(\varphi)$ Rossby Waves on a Sphere

		m			
		0	1	2	3
	1	$\sin\varphi$	$\cos\varphi$	–	–
n	2	$\frac{1}{2}(3\sin^2\varphi - 1)$	$-3\sin\varphi\cos\varphi$	$3\cos^2\varphi$	–
	3	$\frac{3}{2}\sin\varphi(5\sin^2\varphi - 3)$	$-\frac{9}{2}(5\sin^2\varphi - 1)\cos\varphi$	$45\sin\varphi\cos^3\varphi$	$-45\cos^3\varphi$

where P_m^n is an associated Legendre function of degree n and order m, with $n > m$. The angular frequency must satisfy

$$\sigma = \frac{-2\Omega}{n(n+1)}. \tag{16.10}$$

As in the case of barotropic Rossby waves in a fluid at rest on a β plane, spherical Rossby waves propagate westward. Their zonal phase speed is given by

$$c = a\cos\varphi\sigma = -2\Omega a\frac{\cos\varphi}{n(n+1)}. \tag{16.11}$$

The first few associated Legendre functions are given in Table 16.1. The lowest order modes, for which $m = 0$, are zonally symmetric and have zero frequency. These are just east-west flows that do not perturb the background vorticity gradient and thus are not oscillatory. The lowest order wave mode, for which $n = m = 1$, has an angular frequency of $-\Omega$ and is therefore stationary relative to absolute space. This zonal wavenumber 1 mode has maximum amplitude on the equator and decays as $\cos\varphi$ toward the poles. Modes of greater values of n have increasingly fine meridional structure.

17. Quasi-geostrophic Rossby waves

Baroclinic flows can also support Rossby wave propagation. This is most easily described using quasi-geostrophic theory. We begin by looking at the behavior of small perturbations to a zonal background flow that varies only in the meridional and vertical directions. Beginning with the definition of pseudo-potential vorticity (9.11), we let φ and q_p be represented by zonally invariant background fields plus perturbations to them:

$$\begin{aligned} \varphi &= \overline{\varphi}(y,p) + \varphi'(x,y,p,t) \\ q_p &= \overline{q}_p(y,p) + q_p'(x,y,p,t) \end{aligned} \tag{17.1}$$

We next linearize the adiabatic, frictionless form of the conservation equation for q_p (9.10) using (17.1):

$$\frac{\partial q_p'}{\partial t} + \overline{u}_g \frac{\partial q_p'}{\partial x} + v_g' \frac{\partial \overline{q}_p}{\partial y} = 0, \tag{17.2}$$

where

$$\begin{aligned} \overline{u}_g &= -\frac{1}{f}\frac{\partial \overline{\varphi}}{\partial y}, \\ v_g' &= \frac{1}{f}\frac{\partial \varphi'}{\partial x}. \end{aligned} \tag{17.3}$$

We also note from the definition of pseudo-potential vorticity (9.11), that

$$\begin{aligned} \frac{\partial \overline{q}_p}{\partial y} &= \frac{1}{f_0}\nabla^2 \frac{\partial \overline{\varphi}}{\partial y} + \beta_0 + \frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial}{\partial p}\frac{\partial \overline{\varphi}}{\partial y} \\ &= \beta - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial p}\frac{f_0^2}{\mathcal{S}}\frac{\partial \overline{u}}{\partial p}. \end{aligned} \tag{17.4}$$

Thus the meridional gradient of background pseudo-potential vorticity depends on β, the meridional gradient of the vorticity of the zonal wind, and a measure of the curvature of the vertical profile of the mean zonal wind.

Using the second of the geostrophic relations in (17.3) as well as the definition of pseudo-potential vorticity, (9.11), the linearized conservation relation (17.2) may be written

$$\left(\frac{\partial}{\partial t} + \overline{u}_g \frac{\partial}{\partial x}\right)\left[\frac{1}{f_0}\nabla^2\varphi' + \frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}\right] + \frac{1}{f_0}\frac{\partial \overline{q}_p}{\partial y}\frac{\partial \varphi'}{\partial x} = 0. \tag{17.5}$$

We will examine solutions of (17.5) in the special case that the stratification is constant, $\mathcal{S} =$ constant. We will also assume that the background zonal wind, $\overline{u}_g$, and the associated gradient of background pseudo-potential vorticity given by (17.4) are slowly varying compared to the structures of perturbations to the flow. If this is the case, we can make the W.K.B. approximation and represent *modal* solutions to (17.5) as

$$\varphi' = \Phi e^{ik(x-ct)+i\int^y l(y',p)dy' + i\int^p m(y,p')dp'}, \tag{17.6}$$

where $l(y,\rho)$ and $m(y,p)$ are slowly varying functions of latitude and pressure. Substituting (17.6) into (17.5) gives a dispersion relation:

$$c = \overline{u}_g - \frac{\partial \overline{q}_p/\partial y}{k^2 + l^2 + \frac{f_0^2}{\mathcal{S}}m^2}. \tag{17.7}$$

Comparing this to the strictly barotropic dispersion relation (10.8) shows the strong similarity between barotropic and baroclinic waves. The main differences are that in the baroclinic case, the meridional gradient of potential (rather than actual) vorticity serves as the refractive index for Rossby waves, and the vertical structure contributes to the dispersion properties of the waves.

The wave frequency, which remains invariant along the ray paths followed by the wave energy as long as the background flow is considered to be steady, is given by

$$\omega = kc = k\overline{u}_g - \frac{k(\partial \overline{q}_p/\partial y)}{k^2 + l^2 + \frac{f_0^2}{\mathcal{S}} m^2}. \tag{17.8}$$

Letting $k_i \equiv (k, l, m)$, the three components of the group velocity are given by

$$c_{g_i} = \frac{\partial \omega}{\partial k_i} = \left[\left(\overline{u}_g + \frac{\overline{q}_{py}}{r^4} \left(k^2 - l^2 - \frac{f_0^2}{\mathcal{S}} m^2 \right) \right), \frac{2kl\overline{q}_{py}}{r^4}, \frac{2km\frac{f_0^2}{\mathcal{S}}\overline{q}_{py}}{r^4} \right], \tag{17.9}$$

where

$$\overline{q}_{py} \equiv \frac{\partial \overline{q}_p}{\partial y},$$
$$r^2 \equiv k^2 + l^2 + \frac{f_0^2}{\mathcal{S}} m^2.$$

It is of some interest to compare these group velocities to the phase speeds, which are given by

$$\begin{aligned} c_r \equiv \frac{\omega}{k_i} &= \left[\left(\overline{u}_g - \frac{\overline{q}_{py}}{r^2} \right), -\frac{k}{l}\frac{\overline{q}_{py}}{r^2}, -\frac{k}{m}\frac{\overline{q}_{py}}{r^2} \right] \\ &= \left[\left(c_{g_x} - 2\frac{k^2\overline{q}_{py}}{r^4} \right), -\frac{1}{2}\frac{r^2}{l^2}c_{gy}, -\frac{1}{2}\frac{\mathcal{S}}{f_0^2}\frac{r^2}{m^2}c_{gp} \right] \end{aligned} \tag{17.10}$$

Thus, for quasi-geostrophic Rossby waves, the flow-relative group velocity in the meridional and vertical directions is of opposite sign from the phase speeds in those directions.

As quasi-geostrophic Rossby waves disperse in three dimensions, the associated wave numbers evolve following the vector group velocity, according to the relationship for the refraction of wave energy:

$$\frac{dk_i}{dt} = -\frac{\partial \omega}{\partial x_i}, \tag{17.11}$$

where the total derivative indicates the rate of change following the group velocity:

$$\frac{dk_i}{dt} = \frac{\partial k_i}{\partial t} + c_{g_j} \frac{\partial k_i}{\partial x_j}.$$

Using (17.8), the evolution of wavenumber (17.11) is

$$\frac{dk_i}{dt} = \left[0, k\left(-\frac{\partial \overline{u}_g}{\partial y} + \frac{\partial^2 \overline{q}_p/\partial y^2}{r^2} \right), k\left(-\frac{\partial \overline{u}_g}{\partial \rho} + \frac{\partial^2 \overline{q}_p/\partial y \partial p}{r^2} \right) \right]. \tag{17.12}$$

The interaction between Rossby waves and the background flow is of great interest, because in quasi-balanced flows these waves are responsible for conveying information from one place to another. An elegant way of quantifying the interaction between quasi-geostrophic Rossby waves and the background state on which they are assumed to propagate is through the examination of *Eliassen-Palm fluxes.* The *Eliassen-Palm theory* is derived as follows.

Since quasi-geostrophic flow on an f plane is nondivergent, we may write the conservation equation for pseudo-potential vorticity, (9.10), in the form

$$\frac{\partial q_p}{\partial t} = -\frac{\partial}{\partial x}(u_g q_p) - \frac{\partial}{\partial y}(v_g q_p). \tag{17.13}$$

Consider now the time rate of change of zonal mean pseudo-potential vorticity. First define a zonal average operator $\{\quad\}$, such that for any scalar A,

$$\{A\} \equiv \frac{1}{L} \int_0^L A \, dx, \tag{17.14}$$

where L is the distance around a latitude circle. Applying this operator to (17.13) gives

$$\frac{\partial}{\partial t}\{q_p\} = -\frac{\partial}{\partial y}\{v_g q_p\}. \tag{17.15}$$

Now let

$$v_g = \{v_g\} + v_g',$$

$$q_p = \{q_p\} + q_p',$$

where v_g' is the local, instantaneous departure of v_g from $\{v_g\}$, but since

$$v_g = \frac{1}{f_0}\frac{\partial \varphi}{\partial x},$$

$\{v_g\} = 0$. Thus (17.15) becomes

$$\frac{\partial}{\partial t}\{q_p\} = -\frac{\partial}{\partial y}\{v_g' q_p'\}. \tag{17.16}$$

The time rate of change of zonal mean pseudo-potential vorticity is equal to the convergence of the meridional eddy flux of pseudo-potential vorticity.

Using the definitions of q_p and v_g,

$$q_p' = \frac{1}{f_0}\nabla^2\varphi' + \frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial \varphi'}{\partial p},$$
$$v_g' = \frac{1}{f_0}\frac{\partial \varphi'}{\partial x},$$

where φ' is the departure of φ from its zonal average, we can write

$$\begin{aligned} v_g' q_p' &= \frac{1}{f_0^2}\left[\frac{\partial \varphi'}{\partial x}\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial \varphi'}{\partial x}\frac{\partial^2 \varphi'}{\partial y^2}\right] + \frac{\partial \varphi'}{\partial x}\frac{\partial}{\partial p}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p} \\ &= \frac{1}{f_0^2}\left[\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\partial \varphi'}{\partial x}\frac{\partial \varphi'}{\partial y}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial y}\right)^2\right] \\ &\quad + \frac{\partial}{\partial p}\left(\frac{\partial \varphi'}{\partial x}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}\right) - \frac{1}{2\mathcal{S}}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial p}\right)^2. \end{aligned} \tag{17.17}$$

Taking the zonal average of this gives

$$\{v_g' q_p'\} = \frac{1}{f_0^2}\frac{\partial}{\partial y}\left\{\frac{\partial \varphi'}{\partial x}\frac{\partial \varphi'}{\partial y}\right\} + \frac{\partial}{\partial p}\left\{\frac{\partial \varphi'}{\partial x}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}\right\}. \tag{17.18}$$

Using the geostrophic relations and the hydrostatic relation (15.8), this may be written

$$
\begin{aligned}
\{v_g' q_p'\} &= -\frac{\partial}{\partial y}\{u_g' v_g'\} - \frac{\partial}{\partial p}\left\{\frac{f_0}{\mathcal{S}\pi} v_g' \theta'\right\} \\
&\equiv \nabla \cdot \mathbf{F},
\end{aligned}
\tag{17.19}
$$

where $\mathbf{F}$ is the *Eliassen-Palm flux*, given by

$$
\mathbf{F} \equiv -\{u_g' v_g'\}\hat{\jmath} - \left\{\frac{f_0}{\mathcal{S}\pi} v_g' \theta'\right\}\hat{p},
\tag{17.20}
$$

with $\hat{\jmath}$ and $\hat{p}$ unit vectors in y and p. Thus the northward component of the Eliassen-Palm flux is the geostrophic northward eddy flux of zonal (geostrophic) momentum, while the vertical component of the EP flux is the geostrophic northward eddy heat flux.

The utility of the Eliassen-Palm flux lies in its role as a source of *wave activity.* This is a measure of the variance of pseudo-potential vorticity, and is defined as

$$
\mathcal{A} \equiv \frac{1}{2}\frac{q_p'^2}{\overline{q}_{py}}.
\tag{17.21}
$$

We can form an equation for wave activity by multiplying (17.2), modified to take into account dissipation, by q_p' and taking the zonal average of the result:

$$
\frac{\partial}{\partial t}\frac{1}{2}\{q_p'^2\} + \frac{\partial \overline{q}_p}{\partial y}\{v' q_p'\} = \{\mathrm{D} q_p'\}.
\tag{17.22}
$$

Now letting

$$
\overline{q}_{py} \equiv \frac{\partial \overline{q}_p}{\partial y},
$$

and noting that $\overline{q}_{py}$ is not a function of time or longitude, divide (17.22) through by $\overline{q}_{py}$:

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{q_p'^2}{\overline{q}_{py}}\right\} + \{v' q_p'\} = \left\{\mathrm{D}\frac{q_p'}{\overline{q}_{py}}\right\},$$

or using (17.21) and (17.19),

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \mathbf{F} = \mathcal{D}, \tag{17.23}$$

where

$$\mathcal{D} \equiv \left\{\mathrm{D}\frac{q_p'}{\overline{q}_{py}}\right\}.$$

In the absence of dissipation of pseudo-potential vorticity, the rate of change of wave activity is proportional to the divergence of the Eliassen-Palm flux. Conversely, in a steady flow, creation or dissipation of wave activity is signified by a nonzero divergence of the Eliassen-Palm flux.

In the case of plane waves, the Eliassen-Palm flux may be interpreted as the flux of wave activity along wave ray paths, traveling at the group velocity. This is shown as follows:

First, using the definitions of wave activity (17.21) and pseudo-potential vorticity (9.11) and the modal decomposition (17.6), we have

$$\begin{aligned} \mathcal{A} &= \frac{1}{2\overline{q}_{py}}\left\{\left[\frac{1}{f_0}(k^2+l^2) + \frac{f_0}{\mathcal{S}}m^2\right]^2 \Phi^2 e^{2ik(x-ct)+2i\int^y l\,dy' + 2i\int^p m\,dp'}\right\} \\ &= \frac{1}{4\overline{q}_{py}}\Phi^2\left[\frac{1}{f_0}(k^2+l^2) + \frac{f_0}{\mathcal{S}}m^2\right]^2 e^{2i\int^y l\,dy' + 2i\int^p m\,dp'}. \end{aligned} \tag{17.24}$$

On the other hand, using (17.6) in the definition of the Eliassen-Palm flux vector, (17.20), together with the usual geostrophic and hydrostatic relations, gives

$$\mathbf{F} = \left[\frac{1}{2f_0^2} kl\Phi^2 e^{2i\int^y l\,dy' + 2i\int^p m\,dp'}\right]\hat{j} + \left[\frac{1}{2}\frac{mk}{\mathcal{S}}\Phi^2 e^{2i\int^y l\,dy' + 2i\int^p m\,dp}\right]\hat{p}. \quad (17.25)$$

Now comparing (17.25) to (17.24), and using the group velocity relations (17.9) together with the definition of wave activity, (17.25), shows that

$$F_i = c_{g_i}\mathcal{A}. \quad (17.26)$$

Thus, for individual plane waves, the Eliassen-Palm flux is just the product of the Rossby wave group velocity and the wave activity. This does not hold for disturbances consisting of more than one plane wave, or nonmodal disturbances, as will be discussed in Chapter (?). Later on, we will find that the Eliassen-Palm flux is very useful for diagnosing the sources and sinks of Rossby waves from atmospheric observations.

18. Barotropic instability

Consider an inviscid barotropic flow governed by the barotropic vorticity equation

$$\frac{d\eta}{dt} = 0, \tag{18.1}$$

where

$$\eta = \nabla^2 \psi \tag{18.2}$$

and ψ is the streamfunction. There exists a class of exact solutions of (18.1) characterized by

$$\psi = \psi(y),$$
$$\eta = \frac{d^2\psi}{dy^2}.$$

These are just zonal flows that vary meridionally. We have seen that flows with cross-stream variations of η can support Rossby waves. Now consider a class of jet-like flows that look like the example shown in Figure 18.1, where there is an extremum in the vorticity. In the example given, there is a maximum of vorticity in the center of the domain, with westward flow to the north and eastward flow to the south. The meridional gradient of vorticity is negative to the north of the vorticity maximum, and positive to the south, so that barotropic Rossby waves propagate eastward, relative to the flow, to the north; and westward, relative to the flow, to the south.

Now consider perturbing the flow in Figure 18.1 in the manner shown in Figure 18.2:

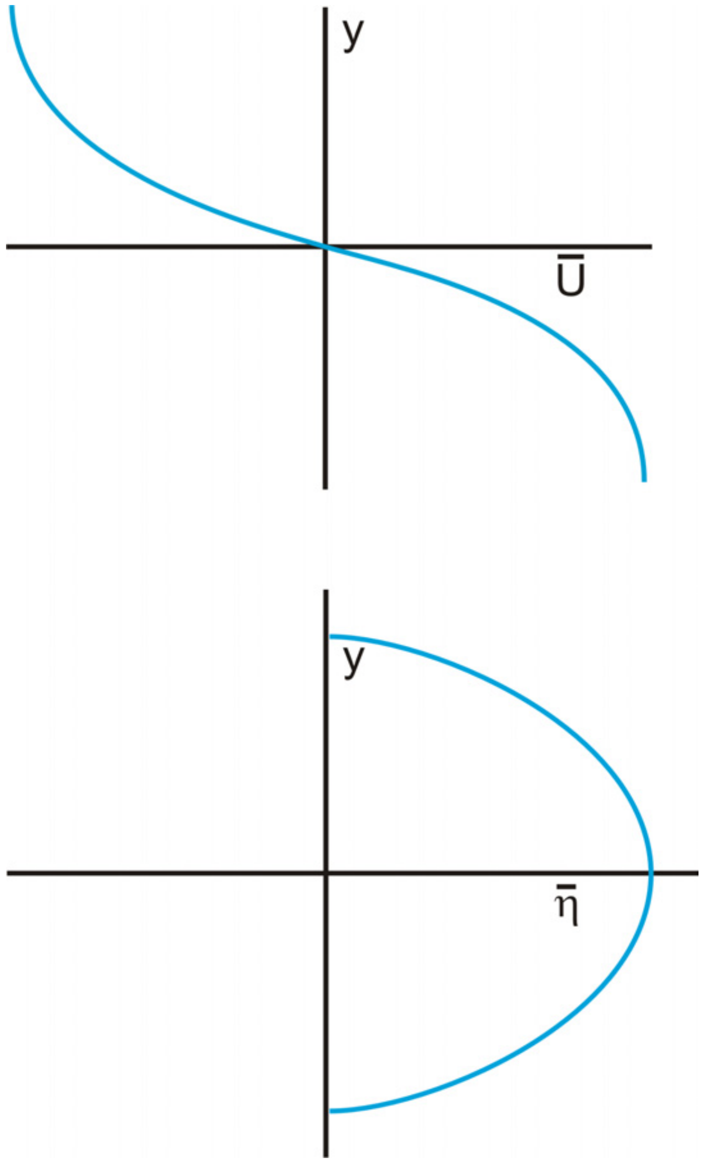

Figure 18.1

This particular deformation of the background vorticity contours produces positive perturbations of vorticity along a southwest-northeast axis. The sense of the perturbation flow associated with these vorticity anomalies is also illustrated in Figure 18.2. Note the follow major points:

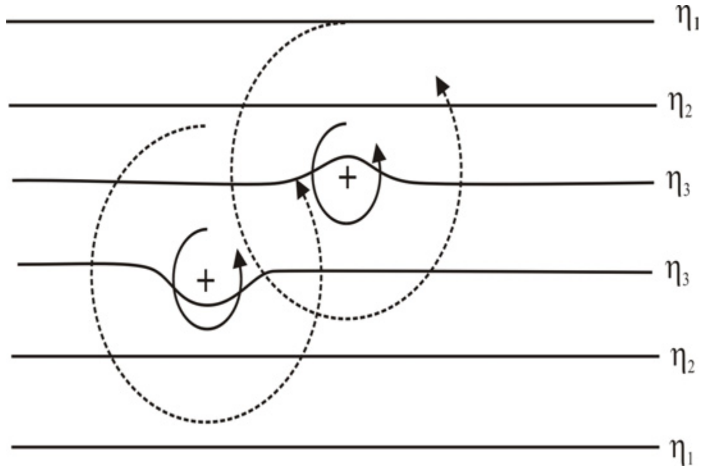

Figure 18.2

1. The local advection of the background gradient of vorticity by the flow associated with the northern perturbation is such as to propagate it eastward *relative to the background flow,* which is toward the west. Thus the intrinsic propagation is opposite to the background flow.

2. The local advection of the background vorticity by the flow associated with the southern vorticity anomaly is such as to propagate the anomaly westward *relative to the background flow,* which is toward the east. Again, the intrinsic propagation is opposite to the background flow.

3. In the vicinity of the *southern* vorticity anomaly, the advection of the background vorticity by the flow associated with the *northern* vorticity anomaly is such as to amplify the southern vorticity anomaly. The same goes for the northern vorticity anomaly.

There are two critical aspects of this scenario. The first is that the difference in the intrinsic phase speeds of the two anomalies is compensated by the different advections of the anomalies by the background flow, creating the possibility that the anomalies can be *phase locked* with one another. The second is that the anomalies can be *mutually amplifying;* i.e., each anomaly amplifies the other anomaly. These two aspects are critical to the process called *barotropic instability.* In the following section, we find analytic solutions to a particular example of a barotropically unstable flow.

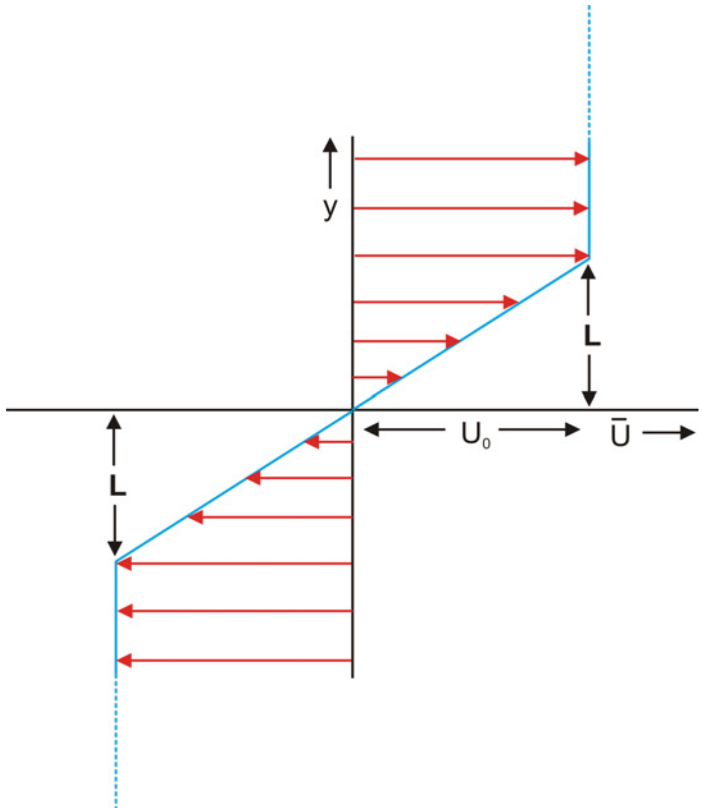

Figure 18.3

18.1 the Rayleigh problem

Consider the barotropic flow whose variation in y is illustrated in Figure 18.3.

The flow is piecewise continuous and all the meridional gradient of vorticity is concentrated in positive and negative delta functions at $y = +D$ and $y = -D$, respectively. We will formulate linear equations for small perturbations to this background state, solve them in each of the three regions, and match the solutions across the boundaries of the regions at $y = \pm D$.

The linearized momentum equations and mass continuity are as follows. Over-bars signify the background state shown in Figure 18.3.

$$\frac{\partial u}{\partial t} + \overline{U}\frac{\partial u}{\partial x} + v\frac{\partial \overline{U}}{\partial y} = -\alpha_0 \frac{\partial p}{\partial x}, \tag{18.1}$$

$$\frac{\partial v}{\partial t} + \overline{U}\frac{\partial v}{\partial x} = -\alpha_0 \frac{\partial p}{\partial y}, \tag{18.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{18.3}$$

Consider solutions of the form

$$u = \tilde{u}(y)e^{ik(x-ct)},$$

$$v = \tilde{v}(y)e^{ik(x-ct)},$$

$$p = \tilde{p}(y)e^{ik(x-ct)},$$

where c may be complex. Substitution into (18.1)–(18.3) gives

$$ik(\overline{U} - c)\tilde{u} + \overline{U}_y\tilde{v} = -\alpha_0 ik\tilde{p}, \qquad (18.4),$$

$$ik(\overline{U} - c)\tilde{v} = -\alpha_0 \frac{d\tilde{p}}{dy}, \tag{18.5}$$

$$ik\tilde{u} + \frac{d\tilde{v}}{dy} = 0. \tag{18.6}$$

By cross-differentiating, we can eliminate $\tilde{u}$ and $\tilde{p}$ from (18.4)–(18.6) to arrive at a single O.D.E. in $\tilde{v}$:

$$\frac{d^2\tilde{v}}{dy^2} - \tilde{v}\left(k^2 + \frac{\overline{U}_{yy}}{\overline{U} - c}\right) = 0. \tag{18.7}$$

Now note that in the interiors of each of the three regions in Figure 18.1, $\overline{U}_{yy} = 0$, so (18.7) reduces to

$$\frac{d^2\tilde{v}}{dy^2} - k^2\tilde{v} = 0 \tag{18.8}$$

within each region. Now we consider boundary conditions for solving (18.8). First we impose the condition that the solutions remain bounded at $y = \pm\infty$:

$$\lim_{y \to \pm\infty} \tilde{v} = \text{ finite.}$$

General solutions of (18.8) that satisfy this condition are:

$$\begin{aligned} \text{I:} \qquad & \tilde{v} = Ae^{-ky} \\ \text{II:} \qquad & \tilde{v} = Be^{-ky} + Ce^{ky} \\ \text{III:} \qquad & \tilde{v} = Fe^{ky} \end{aligned} \tag{18.9}$$

Next we apply boundary conditions at the boundaries separating the regions. There are two fundamental requirements:

a. Fluid displacements must be continuous, and

b. Pressure must be continuous.

The displacement in y, δy, is related to v by

$$v \equiv \frac{d}{dt}\delta y.$$

Linearizing this about the background state gives

$$v = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)\delta y,$$

and substituting

$$\delta y = \tilde{\delta} y e^{ik(x-ct)},$$

we have

$$\tilde{v} = ik(\overline{U} - c)\tilde{\delta} y.$$

Continuity of $\tilde{\delta y}$ demands that

$$\frac{\tilde{v}}{ik(\overline{U}-c)} \text{ is continuous.} \tag{18.10}$$

In the present case, $\overline{U}$ itself is continuous, so (18.10) implies that $\tilde{v}$ is continuous.

Continuity of pressure implies, through (18.4), that the quantity

$$ik(\overline{U}-c)\tilde{u} + \overline{U}_y \tilde{v}$$

is continuous, or using (18.6) to eliminate $ik\tilde{u}$,

$$(\overline{U}-c)\frac{d\tilde{v}}{dy} - \overline{U}_y \tilde{v} \qquad \text{is continuous.} \tag{18.11}$$

Matching $\tilde{v}$ and the quantity given by (18.11) across each of the two boundaries in the general solutions (18.9) gives a condition on the relation between the complex phase speed c and k:

$$\left(\frac{Dkc}{U_0}\right)^2 = (Dk-1)^2 - e^{-2kD}. \tag{18.12}$$

Here we see that c is purely real if $(Dk-1)^2 \geq e^{-2kD}$, and purely imaginary if $(Dk-1)^2 < e^{-2kD}$. In the former case, examination of (18.12) shows that to order $1/k$,

$$\lim_{k\to\infty} c = \pm U_0 \left[1 - \frac{2}{kD}\right]. \tag{18.13}$$

Very small-scale perturbations are confined to the delta function vorticity gradients at $y = \pm\frac{1}{2}D$ and move with the mean flow speed at the respective boundaries between regions, slightly slowed down owing the Rossby propagation effect.

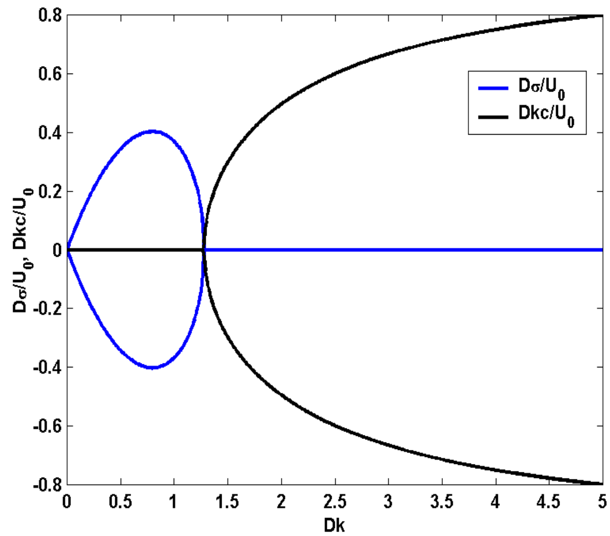

Figure 18.4

In the opposite limit of small k, we have to expand (18.12) to order k^2 to get

$$\lim_{k\to 0} c = \pm i U_0.$$

This denotes perturbations that do not propagate but which grow or decay at an exponential rate given by

$$\sigma \equiv k c_i,$$

so that

$$\lim_{k\to 0} \sigma = \pm k U_0. \tag{18.14}$$

For small k, the growth rate increases linearly with k.

Exact solutions of (18.12) are displayed in Figure 18.4.

Note the following:

1. $c = \sigma = 0$ when $Dk = 1.278$, corresponding to a wavelength $L \equiv \frac{2\pi}{k} = 4.92D$.

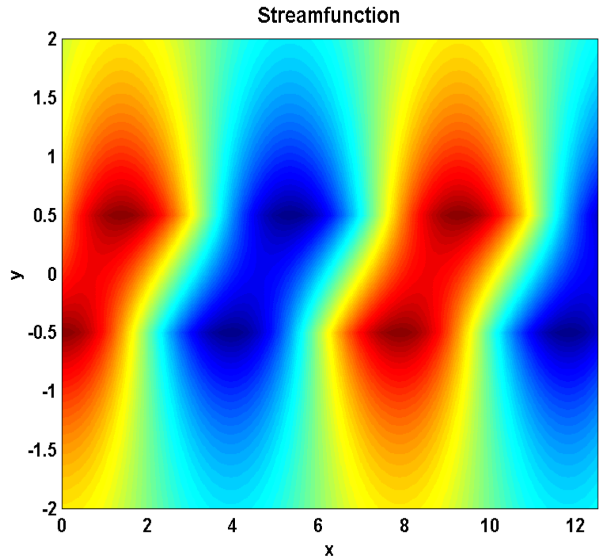

Figure 18.5a

2. The maximum (minimum) value of σ is $\pm 0.402 U_0/D$ occurring when $Dk = 0.797$ ($L = 7.88D$).

Thus modal solutions fall into two classes: long waves that are stationary (and phase locked) and are either amplifying or decaying, and short neutral Rossby waves swimming upstream. These waves are too short to "feel" each other enough to become phase locked or mutually amplifying (or decaying).

The eigenmodes of the velocity streamfunction are shown in Figure 18.5 for $Dk = 0.797$, corresponding to the most rapidly growing mode, and for $Dk = 2.0$, corresponding to a stable mode on the southern vorticity gradient delta function. Note the "upshear" tilt of the unstable mode; eigenfunctions of the decaying mode tilt downshear.

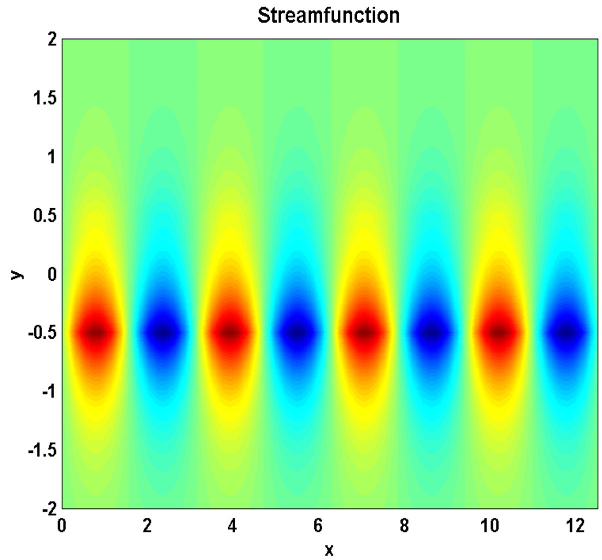

Figure 18.5b

18.2 Necessary conditions for barotropic instability

Rayleigh developed some general necessary conditions for instability of barotropic flows. These can also be stated as sufficient conditions for stability.

First suppose we have a barotropic flow $\overline{U}(y)$ in which the vorticity gradient is confined to some finite region, so that

$$\lim_{y \to \pm\infty} \frac{d^2\overline{U}}{dy^2} = 0.$$

For such a flow, we expect perturbations to vanish at $y = \pm\infty$ since the refractive index for wave propagation, the vorticity gradient, vanishes there.

Now consider modal disturbances to such a flow. These are governed by (18.7), for the meridional structure of the meridional wind. Now multiply (18.7) through by the complex conjugate of $\tilde{v}$, $\tilde{v}^*$, and integrate the result through the whole domain:

$$\int_{-\infty}^{\infty}\left[\tilde{v}^*\frac{d^2\tilde{v}}{dy^2}-|\tilde{v}|^2\left(k^2+\frac{\overline{U}_{yy}}{\overline{U}-c}\right)\right]dy=0 \tag{18.15}$$

Here we have made use of the fact that

$$\tilde{v}\tilde{v}^*=|\tilde{v}|^2,$$

where $|\tilde{v}|$ is the absolute value of $\tilde{v}$. Now the first term in the integrand can be integrated by parts:

$$\int_{-\infty}^{\infty}\tilde{v}^*\frac{d^2\tilde{v}}{dy^2}dy=\int_{-\infty}^{\infty}\frac{d}{dy}\left[\tilde{v}^*\frac{d\tilde{v}}{dy}\right]dy-\int_{-\infty}^{\infty}\left|\frac{d\tilde{v}}{dy}\right|^2dy.$$

The first term on the right can be integrated exactly, but it vanishes because $\tilde{v}\to 0$ as $y\to\pm\infty$. Thus

$$\int_{-\infty}^{\infty}\tilde{v}\frac{d^2\tilde{v}}{dy^2}dy=-\int_{-\infty}^{\infty}\left|\frac{d\tilde{v}}{dy}\right|^2dy. \tag{18.16}$$

Using (18.16), we may write (18.15) as

$$\int_{-\infty}^{\infty}\left[\left|\frac{d\tilde{v}}{dy}\right|^2+|\tilde{v}|^2\left(k^2+\frac{\overline{U}_{yy}}{\overline{U}-c}\right)\right]dy=0. \tag{18.17}$$

Remember that c is, in general, complex, so the real and imaginary parts of (18.17) must both be satisfied. The imaginary part of (18.17) is

$$c_i\int_{-\infty}^{\infty}\frac{\overline{U}_{yy}}{|\overline{U}-c|^2}|\tilde{v}|^2dy=0, \tag{18.18}$$

where c_i is the imaginary part of c, which is positive for growing disturbances.

The relation (18.18) shows that one of two things must be true: Either

a. $c_i = 0$, or

b. the integral in (18.18) vanishes.

Thus we may conclude the following:

1. A *necessary* condition for instability ($c_i > 0$) is that $\overline{U}_{yy}$ change sign at least once within the domain. In other words, the mean state vorticity must have an extremum in the domain. But note that even if $\overline{U}_{yy}$ does change sign, this is no *guarantee* that the integral vanishes or that $c_i > 0$. This condition is *not* sufficient for instability.

2. If there is no extremum of vorticity within the domain, $c_i = 0$ and this is therefore a *sufficient* condition for stability.

Points 1 and 2 are really saying the same thing.

Another theorem, due to Fjøtoft, may be derived by looking at the real part of (18.17):

$$\int_{-\infty}^{\infty} \frac{\overline{U}_{yy}(\overline{U} - c_r)}{|\overline{U} - c|^2} |\tilde{v}|^2 dy = - \int_{-\infty}^{\infty} \left[\left| \frac{d\tilde{v}}{dy} \right|^2 + k^2 |\tilde{v}|^2 \right] dy, \tag{18.19}$$

where c_r is the real part of c. Note that for *growing* disturbances, we are free to add any multiple of the integral in (18.18) to the left side of (18.19), since the former vanishes. We choose the multiplying factor to be c_r, giving

$$\int_{-\infty}^{\infty} \frac{\overline{U}\overline{U}_{yy}}{|\overline{U} - c|^2} |\tilde{v}|^2 dy = - \int_{-\infty}^{\infty} \left[\left| \frac{d\tilde{v}}{dy} \right|^2 + k^2 |\tilde{v}|^2 \right] dy. \tag{18.20}$$

Since the right-hand side of (18.20) is negative definite, so must the left side. So fluctuations of $\overline{U}$ *must be negatively correlated with* $\overline{U}_{yy}$ for growing disturbances.

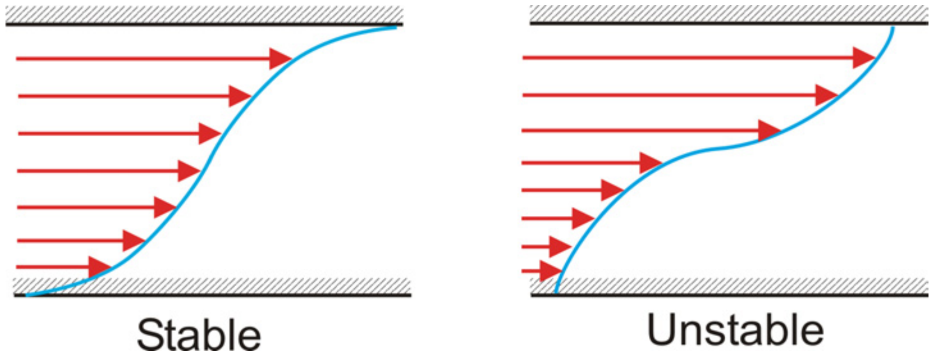

Figure 18.6

(Again, this is a necessary but not sufficient condition for instability.) Figure 18.6 shows an example of a flow that the Fjørtoft theorem shows to be stable in spite of satisfying the Rayleigh necessary condition for instability, and a flow which satisfies both necessary conditions for instability.

19. Baroclinic Instability

In two-dimensional barotropic flow, there is an exact relationship between mass streamfunction ψ and the conserved quantity, vorticity (η) given by $\eta = \nabla^2\psi$. The evolution of the conserved variable η in turn depends only on the spatial distribution of η and on the flow, which is derivable from ψ and thus, by inverting the elliptic relation, from η itself. This strongly constrains the flow evolution and allows one to think about the flow by following η around and inverting its distribution to get the flow.

In three-dimensional flow, the vorticity is a vector and is not in general conserved. The appropriate conserved variable is the potential vorticity, but this is not in general invertible to find the flow, unless other constraints are provided. One such constraint is geostrophy, and a simple starting point is the set of quasi-geostrophic equations which yield the conserved and invertible quantity q_p, the pseudo-potential vorticity.

The same dynamical processes that yield stable and unstable Rossby waves in two-dimensional flow are responsible for waves and instability in three-dimensional baroclinic flow, though unlike the barotropic 2-D case, the three-dimensional dynamics depends on at least an approximate balance between the mass and flow fields.

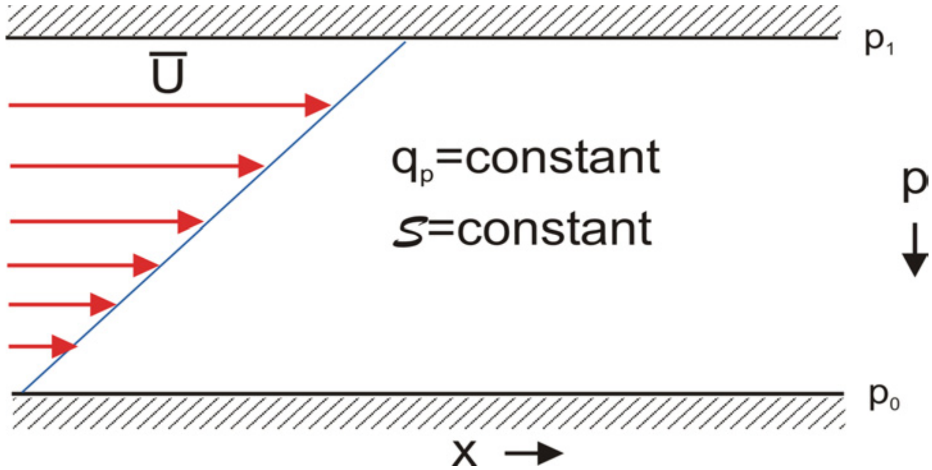

Figure 19.1

a. The Eady model

Perhaps the simplest example of an instability arising from the interaction of Rossby waves in a baroclinic flow is provided by the *Eady Model,* named after the British mathematician Eric Eady, who published his results in 1949. The equilibrium flow in Eady's idealization is illustrated in Figure 19.1. A zonal flow whose velocity increases with altitude is confined between two rigid, horizontal plates. This flow is in exact thermal wind balance with an equatorward-directed temperature gradient and is considered to have constant pseudo potential vorticity, q_p, as well as constant background static stability, $\mathcal{S}$. The flow occurs on an f plane, so $\beta = 0$. Evolution of the flow is taken to be inviscid and adiabatic.

At first blush, it might appear that no interesting quasi-geostrophic dynamics can occur in this system since there are no spatial gradients of q_p and thus no Rossby waves. But there are temperature gradients on both boundaries, and according to the analysis presented in Sections 11 and 12, Eady edge waves—boundary-trapped Rossby waves—can exist. Eady showed that the two sets of Rossby waves corresponding to both boundaries can interact unstably, giving rise to exponential instability.

Since pseudo potential vorticity is conserved in this problem, and since it is initially constant, perturbations to it vanish. According to (9.11),

$$\frac{1}{f_0}\nabla^2\varphi + \frac{f_0}{\mathcal{S}}\frac{\partial^2\varphi}{\delta p^2} = 0, \tag{19.1}$$

which was already derived as (12.1). Here φ is now defined as a perturbation to the background geopotential distribution.

The background zonal flow is specifically defined to be linear in pressure:

$$\frac{d\overline{u}}{dp} = \frac{R}{f_0 p_0}\frac{d\overline{\theta}}{dy} = -\gamma, \tag{19.2}$$

where $\overline{u}$ is the background zonal wind, $\overline{\theta}$ is the background potential temperature, f_0 is the Coriolis parameter and R is the gas constant. We introduce γ for notational convenience. For the ocean, γ would be defined as $-G\overline{\sigma}_y$, evaluated at a suitable pressure level.

Integrating (19.2) in pressure, we get a relationship between the background zonal velocities at the two boundaries:

$$\overline{u}_1 - \overline{u}_0 = \gamma(p_0 - p_1), \tag{19.3}$$

where p_0 and p_1 are the pressures at the two boundaries. The system is Galilean invariant, so we can add an arbitrary constant to the background zonal flow. We choose this so that the background zonal flow at one boundary is equal in magnitude but opposite in sign to the flow at the other boundary, to wit

$$\begin{aligned} u_1 &= \frac{1}{2}\gamma(p_0 - p_1) \equiv \frac{1}{2}\Delta u, \\ u_0 &= -\frac{1}{2}\gamma(p_0 - p_1) \equiv -\frac{1}{2}\Delta u, \end{aligned} \tag{19.4}$$

where Δu is the background shear, $\overline{u}_1 - \overline{u}_0$. From (19.3) we have

$$\Delta u = \gamma \Delta p, \tag{19.5}$$

where $\Delta p \equiv p_0 - p_1$.

To solve (19.1), we need to impose boundary conditions. We take perturbations to the background to be periodic in the two horizontal directions. In the vertical, the appropriate boundary conditions are given by (11.1):

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right)\theta = 0 \text{ on } p = p_0, p_1. \tag{19.6}$$

As in section 12, we *linearize* this boundary condition around the background zonal flow, assuming that perturbations to it are so small that contributions to (19.6) that are quadratic in the perturbations can be neglected. (Note that (19.6) is the *only* equation in Eady's system that is linearized.) Linearization of (19.6) gives

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\theta' + v'\frac{d\overline{\theta}}{dy} = 0 \text{ on } p = p_0, p_1. \tag{19.7}$$

Using the hydrostatic equation for θ', the geostrophic relation for v' and (19.2) for $d\overline{\theta}/dy$ gives

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \varphi}{\partial p} + \gamma\frac{\partial \varphi}{\partial x} = 0 \text{ on } p = p_0, p_1. \tag{19.8}$$

Specializing this to the upper and lower boundaries of the Eady model using (19.4) gives

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta u \frac{\partial}{\partial x}\right)\frac{\partial \varphi}{\partial p} + \gamma \frac{\partial \varphi}{\partial x} &= 0 \text{ on } p = p_1, \\
\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta u \frac{\partial}{\partial x}\right)\frac{\partial \varphi}{\partial p} + \gamma \frac{\partial \varphi}{\partial x} &= 0 \text{ on } p = p_0.
\end{aligned} \tag{19.9}$$

Thus the mathematical problem to be solved is given by (19.1) coupled to (19.9), remembering that we are applying periodic boundary conditions in x and y.

Since (19.1) is a linear elliptic equation with constant coefficients, we look for solutions in terms of exponential normal modes of the form

$$\varphi = [A\sinh(rp) + B\cosh(rp)]e^{ik(x-ct)+ily}, \tag{19.10}$$

where c is a (potentially complex) phase speed, and r, k, and l are wavenumbers in pressure and in x and y, respectively. From (19.1) we have

$$r^2 = \frac{\mathcal{S}}{f_0^2}(k^2 + l^2), \tag{19.11}$$

which shows that the vertical exponential decay scale of the disturbances is related to a measure of the horizontal scale by the Rossby aspect ratio $\frac{f_0}{\sqrt{\mathcal{S}}}$.

It proves convenient to nondimensionalize the complex phase speed, c, and the vertical wavenumber, r, according to

$$\begin{aligned}
c &\to \Delta u c, \\
r &\to r/\Delta p.
\end{aligned} \tag{19.12}$$

Making us of these an substituting (19.10) into the two vertical boundary conditions (19.9) gives the dispersion relation:

$$c^2 = \frac{1}{4} + \frac{1}{r^2} - \frac{ctnh(r)}{r}. \tag{19.13}$$

At first take, it would appear that the time dependence of the normal modes of the Eady problem is independent of the particular values of the horizontal wavenumbers k and l, depending only on their combination $k^2 + l^2$ through (19.11). But if c has a nonzero imaginary part, c_i, then (19.10) shows that the exponential growth rate is given by kc_i, so we shall be concerned about k as well.

Although (19.13) can be easily graphed, it is interesting to explore certain limiting and special cases. In the small horizontal wavelength limit, we have

$$\lim_{r\to\infty} c^2 = \frac{1}{4} + \frac{1}{r^2} - \frac{1}{r} = \left(\frac{1}{2} - \frac{1}{r}\right)^2, \tag{19.14}$$

whose solution is

$$c = \pm\left(\frac{1}{2} - \frac{1}{r}\right). \tag{19.15}$$

These are just the solutions of the Eady edge wave problem solved in section 12, in the limit of large r, with the positive root corresponding to the upper boundary. In nondimensional terms, $\frac{1}{2}$ corresponds to the background flow at the upper boundary, while $-\frac{1}{2}$ corresponds to the background flow at the lower boundary. So these are small, stable Eady waves at each boundary, swimming upstream. This is the same as the asymptotic solution of the Eady edge wave at each boundary independently, given by (12.13), so in this limit, the two edge waves pass each other like ships in the night, ignorant of each other's existence.

In the limit of large wavelength (small r), meaningful solutions require us to expand the $ctnh(r)$ term in (19.13) to second order, to wit

$$\lim_{r\to 0} r\,ctnh(r) = 1 + \frac{1}{2}r^2.$$

This gives

$$\lim_{r\to 0} c^2 = -\frac{1}{4},$$

or

$$\lim_{r\to 0} c = \pm\frac{1}{2}i. \tag{19.16}$$

Substitution into (19.10) shows that these modes have vanishing phase speed but grow or decay at an exponential rate given by

$$\sigma \equiv kc_i = \pm\frac{1}{2}k. \tag{19.17}$$

Thus longwave modes of the Eady model are stationary and grow or decay exponentially in time.

Examination of the dispersion relation (19.13) shows that $c^2 = 0$ for a particular value of r which turns out to be $\simeq 2.4$. Also, in the exponential regime at long wavelength, the quantity c^2r^2 has an extremum when $r = 1.606$ corresponding to a value of rc_i of 0.3098. From (19.11), we have that

$$k = \sqrt{\frac{r_0^2}{\mathcal{S}}r^2 - l^2},$$

so the maximum growth rate, kc_i, is given by

$$(kc_i)_{\max} = 0.3098\sqrt{1 - \frac{l^2}{r^2}},$$

where we have used a suitable nondimensionalization of l. This shows that *the maximum growth rate always occurs for* $l = 0$, i.e., for disturbances that are independent of y.

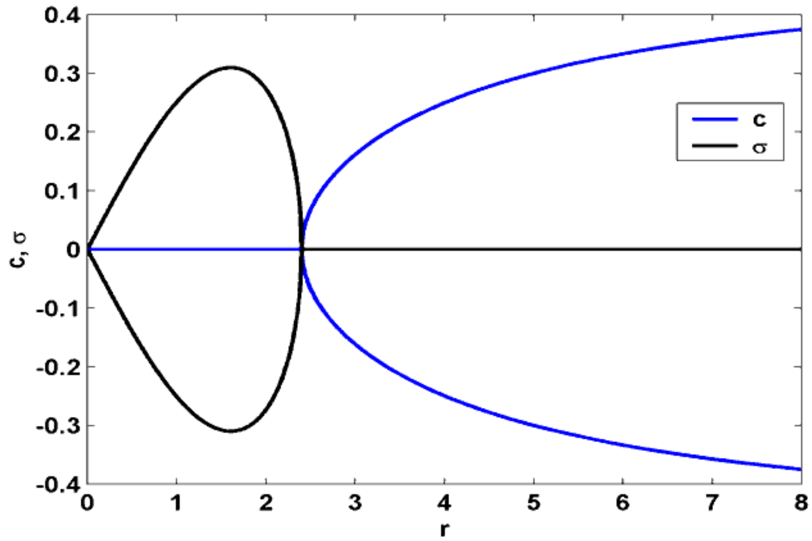

Figure 19.2

The complete solution to (19.13) is graphed in Figure 19.2. For $r < 2.4$, the modes are nonpropagating and exponentially growing or decay. For $r > 2.4$, there are two neutral propagating roots corresponding to eigenfunctions that maximize at one or the other boundary—the two Eady edge waves.

Note that the solutions to the Eady problem in Figure 19.2 closely resemble the solutions of the Rayleigh barotropic instability problem discussed in section 18.1 and shown in Figure 18.4. In fact, *the dynamics are essentially the same;* the only difference is one of geometry: whereas the Rossby waves in the barotropic Rayleigh problem interact laterally, those in the baroclinic Eady problem interact vertically. But *there is no fundamental difference between barotropic and baroclinic instability,* although in the pure barotropic case the disturbance energy is drawn from the kinetic energy of the mean flow, whereas in the pure baroclinic case it is drawn from the potential energy inherent in the background horizontal temperature gradient.

As in Rayleigh's problem, the instability of the Eady basic state can be usefully regarded as resulting from the mutual amplification of phase-locked Rossby waves, as illustrated in Figure 19.3. If a cold anomaly at the upper boundary is positioned west of a warm anomaly at the lower boundary, invertibility gives cyclonic circulation at the location of each of the two boundary temperature anomalies, decaying exponentially away from the boundary. The cyclonic circulation associated with the upper cold anomaly, projected down the lower boundary gives a poleward flow at the location of the lower warm anomaly. Advection of the background temperature gradient leads to a positive temperature tendency there, reinforcing the existing lower boundary temperature anomaly. Likewise, the cyclonic circulation associated with the lower warm anomaly, projecting up to the upper boundary, causes a temperature advection that amplifies the upper cold anomaly. Note, however, that there are small phase shifts between the boundary temperature anomalies and the temperature advection, owing to the circulations induced by the temperature anomalies at the opposite boundaries. These phase shifts serve to alter the propagation speeds of the disturbances, keeping them phase-locked.

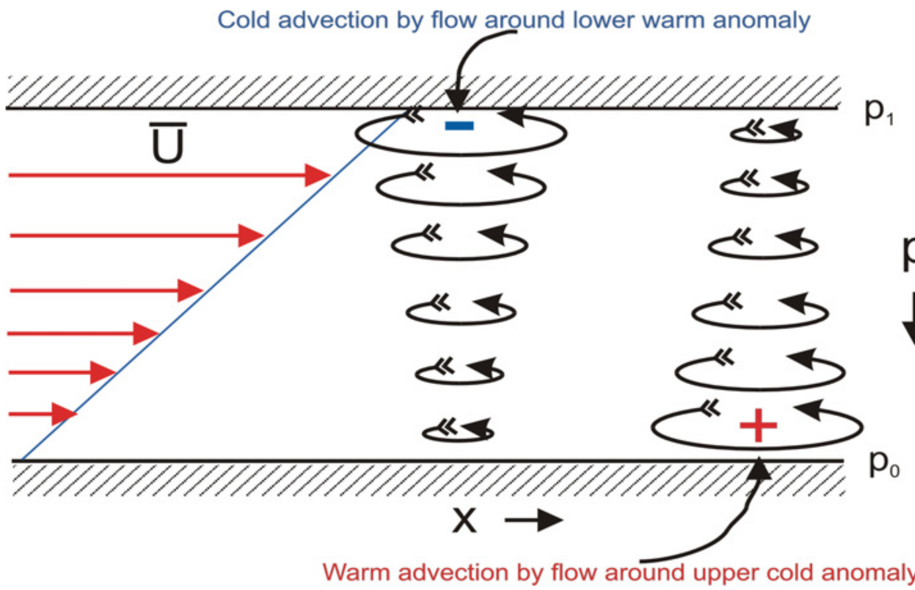

Figure 19.3

b. The Charney Model

At the same time Eady was developing his model of baroclinic instability, Jule Charney, then a graduate student at UCLA, was working on a somewhat different model, which he ultimately published in 1947. Charney used essentially the quasi-geostrophic equations, and took as his basic state one of a zonal wind increasing linearly with altitude, as in the Eady model. But unlike the latter, Charney did not apply an upper lid, allowing his domain to be semi-infinite, and instead of having constant pseudo-potential vorticity, he took the background state to have a constant meridional gradient of q_p. Following Charney's original paper, we here work in height coordinates, rather than pressure coordinates. One can show that in height coordinates, q_p is given by

$$q_p = \frac{1}{f_0}\nabla^2 \frac{p}{\rho_0} + \beta y + \frac{f_0}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{\rho_0 N^2} \frac{\partial p}{\partial z}, \tag{19.18}$$

where p is the perturbation of pressure away from the background state, ρ_0 is a constant reference density, $\overline{\rho}(z)$ is the density distribution of the background state, and

$$N^2 \equiv \frac{g}{\theta_0} \frac{d\overline{\theta}}{dz}, \tag{19.19}$$

where $\overline{\theta}(z)$ is the potential temperature of the background state. Note that N has the units of inverse time and is called the buoyancy frequency, or the Brunt-Vaisälä frequency.

Charney took N^2 = constant and

$$\overline{\rho} = \rho_0 e^{-z/H}, \tag{19.20}$$

with H a (constant) density scale height. Then (19.18) becomes

$$q_p = \frac{1}{f_0}\nabla^2 \frac{p}{\rho_0} + \beta y + \frac{f_0}{N^2 \rho_0} \frac{\partial^2 p}{\partial z^2} - \frac{f_0}{H N^2 \rho_0} \frac{\partial p}{\partial z}. \tag{19.21}$$

As mentioned before, Charney took his basic state q_p to have a constant meridional gradient:

$$\frac{d\overline{q}_p}{dy} = \frac{1}{f_0}\nabla^2 \frac{1}{\rho_0} \frac{d\overline{p}}{dy} + \beta - \frac{f_0}{N^2 \rho_0} \frac{\partial^2}{\partial z^2} \frac{d\overline{p}}{dy} - \frac{f_0}{H N^2 \rho_0} \frac{\partial}{\partial z} \frac{d\overline{p}}{dy}. \tag{19.22}$$

But, using the geostrophic relation

$$\frac{1}{\rho_0} \frac{d\overline{p}}{dy} = -f_0 \overline{u}$$

and remembering that Charney took $\overline{u}$ to be a linear function of z,

$$\frac{d\overline{q}_p}{dy} = \beta + \frac{f_0^2}{HN^2}\frac{d\overline{u}}{dz} = \text{ constant } \equiv \hat{\beta}. \tag{19.23}$$

Here we use $\hat{\beta}$ to denote the constant background pseudo-potential vorticity gradient. We will also define

$$\Lambda \equiv \frac{d\overline{u}}{dz}$$

and take

$$\overline{u} = \Lambda z. \tag{19.24}$$

Linearizing the pseudo-potential vorticity equation (9.10) about this state gives

$$\left(\frac{\partial}{\partial t} + \Lambda z\right) q_p' + v'\hat{\beta} = 0, \tag{19.25}$$

where q' is the perturbation pseudo-potential vorticity, which from (19.21) is given by

$$q_p' = \frac{1}{f_0}\nabla^2\frac{p'}{\rho_0} + \frac{f_0}{N^2\rho_0}\frac{\partial^2 p'}{\partial z^2} - \frac{f_0}{HN^2\rho_0}\frac{\partial p'}{\partial z}. \tag{19.26}$$

Charney's lower boundary condition is identical to Eady's, given that the constant vertical shear of the background zonal wind must be associated with a constant background meridional gradient of potential temperature. Making explicit use of the hydrostatic and thermal wind equation, we have as a lower boundary condition

$$\frac{\partial}{\partial t}\frac{\partial p'}{\partial z} - \Lambda\frac{\partial p'}{\partial x} = 0 \text{ on } z = 0. \tag{19.27}$$

Charney applied a *wave radiation condition* at $z \to \infty$. This asserts that, away from the origin of the waves, the wave energy propagation must be away from the source.

In this case, it implies that wave energy must be travelling upward through the top of the domain. On the other hand, Charney was primarily interested in growing (unstable) disturbances. Such waves should decay exponentially away from their source, so

$$\lim_{z\to\infty} p' = 0, \tag{19.28}$$

which we apply as an upper boundary condition.

Since the coefficients of (19.26), (19.27), and (19.28) are constant in x, y, and time, we can look for normal mode solutions of the form

$$p' = \hat{p}(z)e^{ik(x-ct)+ily}, \tag{19.29}$$

where c is complex.

Substituting into (19.26) and the boundary conditions (19.27) and (19.28) gives

$$\frac{d^2\hat{p}}{dz^2} - \frac{1}{H}\frac{d\hat{p}}{dz} + \frac{N^2}{f_0^2}[\frac{\beta^2}{\Lambda z - c} - k^2 - l^2]\hat{p} = 0, \tag{19.30}$$

$$c\frac{d\hat{p}}{dz} + \Lambda\hat{p} = 0 \text{ on } z = 0, \tag{19.31}$$

and

$$\hat{p} = 0 \text{ on } z = \infty (\text{for } c_i > 0). \tag{19.32}$$

Since (19.3) has a nonconstant coefficient, its solution is not in terms of simple trigometric functions. Nevertheless, it can be put in a canonical form by a suitable substitution of variables, and solutions can be obtained in terms of confluent hypergeometric functions.

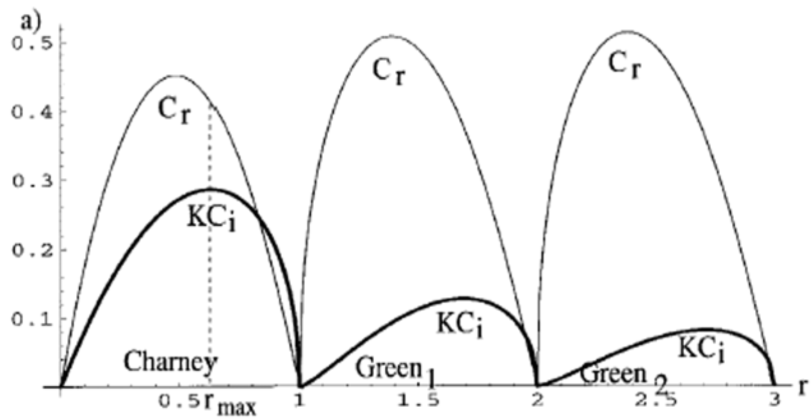

Figure 19.4

An example of solutions for the complex phase speed c is shown in Figure 19.4, while an example of eigenfunctions of unstable modes is shown in Figure 19.5.

Once again, the most unstable modes do not vary with $y(l = 0)$. For the Boussinesq limit ($H \to \infty$), the maximum growth rate is given by

$$\sigma_{\max} = 0.286 \frac{f_0}{N} \Lambda, \tag{19.33}$$

which may be compared to the maximum growth rate in the Eady model of $0.31 \frac{f_0}{N} \Lambda$. This maximum occurs at a horizontal wavenumber $k_{\max}$ whose inverse is given by

$$k_{\max}^{-1} = 1.26 \frac{f_0}{\hat{\beta} N} \Lambda. \tag{19.34}$$

Note that the *maximum* growth rate (19.33) *is independent of* $\hat{\beta}$, but the wavelength of maximum growth is proportional to $\hat{\beta}^{-1}$.

In the Charney model, the surface Eady edge wave, propagating eastward, interacts unstably with an *internal* Rossby wave, living on the background q_p gradient and travelling westward relative to the flow (as opposed to another Eady edge wave, as in the Eady model). As we have seen repeatedly, when looking at the Rayleigh instability problem (section 18) or the Eady problem (section 19a), counter-propagating Rossby waves *must* phase lock for exponential instability. This requirement determines the horizontal and vertical scales of unstable modes in the Charney problem, whereas in the Eady model they are determined by the imposed depth of the system, H.

We can derive the parametric dependence of the wavelength of maximum instability given by (19.34) from the requirement of phase-locking as follows.

First, the Eady edge wave propagates eastward at a rate given approximately by (12.12) and (12.13) which, specialized to the present problem with $l = 0$, is

$$c_{\text{Eady}} \simeq \frac{f_0 \Lambda}{N} \frac{1}{k}. \tag{19.35}$$

Remember that this is just the background zonal wind speed at the altitude of the Rossby penetration depth, f_0/NK. On the other hand, the ground-relative phase speed of a free internal Rossby wave of zonal wavenumber k is

$$c_{\text{Rossby}} \simeq \tilde{u} - \frac{\hat{\beta}}{k^2}, \tag{19.36}$$

where $\tilde{u}$ is some average background wind in the layer containing the Rossby wave. We assume that $\tilde{u}$ scales with the mean wind at the Rossby penetration depth:

$$\tilde{u} \simeq \mu \Lambda \frac{f_0}{Nk},$$

where μ is some number, presumably less than unity, so

$$c_{\text{Rossby}} \simeq \mu \Lambda \frac{f_0}{Nk} - \frac{\hat{\beta}}{k^2}. \tag{19.37}$$

For phase locking, we equate c_{Eady}, given by (19.35), to c_{Rossby}, given by (19.37) to get

$$k^{-1} \simeq (1-\mu)\frac{f_0}{N}\frac{\Lambda}{\hat{\beta}}, \tag{19.38}$$

which is the same scale as (19.34). Thus the most unstable wave has horizontal (and therefore vertical) dimensions that allow it to interact optimally with the surface Eady edge wave.

The vertical scale of the most unstable mode is just the Rossby penetration depth based on the horizontal scale given by (19.34):

$$h \simeq \frac{f_0^2}{N^2}\frac{\Lambda}{\hat{\beta}}.$$

For typical atmospheric values of f_0, N, Λ, and $\hat{\beta}$, this is of order 10 km—curiously close to the actual height of the tropopause.

20. The Charney-Stern Theorem

The Eady problem of baroclinic instability described in section 19a was shown to be remarkably similar to the Rayleigh instability of barotropic flow described in Chapter 18. Both problems can be described in terms of phase-locked, counter-propagating Rossby waves. In section 18.2, we presented Rayleigh's and Fjørtoft's theorems for necessary conditions for the instability of phase-locked barotropic Rossby waves. In 1962, Jule Charney and Melvyn Stern published a generalization of these theorems to the case of three-dimensional, quasi-geostrophic flow.

We begin with equation (9.10) for the conservation of pseudo-potential vorticity, which for an inviscid , adiabatic flow may be written

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla\right)[\frac{1}{f_0}\nabla^2\varphi+\beta y+f_0\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right)]=0. \tag{20.1}$$

Now consider the case of infinitesimal perturbations to a background zonal flow that varies only with latitude and altitude:

$$\begin{aligned}\varphi&=\overline{\varphi}(y,p)+\varphi'(x,y,p,t),\\ \mathbf{V}_g&=\overline{u}(y,p)\hat{i}+\mathbf{V}_g'(x,y,p,t),\\ q_p&=\overline{q}_p(y,p)+q_p'(x,y,p,t).\end{aligned} \tag{20.2}$$

Substituting (20.2) into (20.1) and dropping terms that are quadratic in the perturbation variables gives

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)[\frac{1}{f_0}\nabla^2\varphi'+f_0\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\varphi'}{\partial p}\right)]+v_g'\frac{\partial\overline{q}_p}{\partial y}=0. \tag{20.3}$$

Charney and Stern looked for modal solutions of the form

$$\varphi' = \Phi(y,p)e^{ik(x-ct)}, \tag{20.4}$$

where Φ is a complex function of y and p and c is a complex phase speed. Substitution of (20.4) into (20.3) gives

$$\frac{\partial^2\Phi}{\partial y^2} + f_0^2\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\Phi}{\partial p}\right) + \Phi\left(\frac{\partial\overline{q}_p/\partial y}{\overline{u}-c} - k^2\right) = 0. \tag{20.5}$$

As in section 18.2, we multiply by the complex conjugate of Φ and integrate over a domain that is infinite in y but bounded by rigid plates in p:

$$\int_{-\infty}^{\infty}\int_{p_1}^{p_0}\left[\Phi^*\frac{\partial^2\Phi}{\partial y^2} + f_0^2\Phi^*\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\Phi}{\partial p}\right) + |\Phi|^2\left(\frac{\partial\overline{q}_p}{\partial y}\frac{1}{\overline{u}-c} - k^2\right)\right]dpdy = 0, \tag{20.6}$$

where p_1 and p_0 are the pressures at the top and bottom boundary, respectively. We assume that as $y \to \pm\infty$, the geopotential perturbations or their meridional gradients vanish. Integrating (20.6) by parts and making use of this boundary condition gives

$$\int_{-\infty}^{\infty}\int_{p_1}^{p_0}\left\{\left|\frac{\partial\Phi}{\partial y}\right|^2 + \left(k^2 - \frac{\partial\overline{q}_p/\partial y}{\overline{u}-c}\right)|\Phi|^2 + \frac{f_0^2}{\mathcal{S}}\left|\frac{\partial\Phi}{\partial p}\right|^2\right\}dy\,dp$$
$$+f_0^2\int_{-\infty}^{\infty}\left[\frac{\Phi^*}{\mathcal{S}}\frac{\partial\Phi}{\partial p}\right]\Big|_{p_1}^{p_0}dy = 0. \tag{20.7}$$

The last term in (20.7) involves geopotential perturbations at the two boundaries. For these, we use the Eady boundary condition given by (11.1). Linearizing this about the background state and using (20.4) gives

$$\frac{\partial\Phi}{\partial p} - \frac{\overline{\alpha}}{f_0\overline{\theta}(\overline{u}-c)}\frac{\partial\overline{\theta}}{\partial y}\Phi = 0 \text{ on } p = p_0, p_1. \tag{20.8}$$

Substituting this into the last term of (20.7) gives

$$\int_{-\infty}^{\infty}\left[\int_{p_1}^{p_0}\left(\left|\frac{\partial\Phi}{\partial y}\right|^2+\left(k^2-\frac{\partial\overline{q}_p/\partial y}{\overline{u}-c}\right)|\Phi|^2+\frac{f_0^2}{\mathcal{S}}\left|\frac{\partial\Phi}{\partial p}\right|^2\right)dp\right.$$
$$\left.-\frac{\overline{\alpha}f_0\frac{\partial\overline{\theta}}{\partial y}}{\overline{\theta}\mathcal{S}(\overline{u}-c)}|\Phi|^2\Big|_{p_1}^{p_0}\right]dy=0. \tag{20.9}$$

Since c is in general a complex number, the real and imaginary parts of (20.9) must be satisfied independently. In particular, the imaginary part of (20.9) is

$$c_i\int_{-\infty}^{\infty}\left[\int_{p_1}^{p_0}\left(\frac{\partial\overline{q}_p/\partial y}{|\overline{u}-c|^2}|\Phi|^2\right)dp+\frac{\overline{\alpha}f_0\frac{\partial\overline{\theta}}{\partial y}}{\overline{\theta}\mathcal{S}|\overline{u}-c|^2}|\Phi|^2\Big|_{p_1}^{p_0}\right]dy=0 \tag{20.10}$$

From this expression, it can be seen that for exponentially growing normal modes ($c_i > 0$), one or more of the following must be true:

1. The meridional gradient of pseudo-potential vorticity, $\partial\overline{q}_p\partial y$, changes sign in the domain;
2. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, changes sign along one or both boundaries;
3. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, at the lower boundary (p_0) has the same sign as $\partial\overline{\theta}/\partial y$ at the upper boundary (p_1) and/or the opposite sign of the interior pseudo potential vorticity gradient, $\partial\overline{q}_p/\partial y$;
4. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, at the upper boundary (p_1) has the same sign as either or both the meridional temperature gradient at the lower boundary and the interior potential vorticity gradient.

In the Eady model, $\partial\overline{q}_p/\partial y = 0$ but the temperature gradient has the same sign at both boundaries, so the Charney-Stern necessary condition is satisfied. In the

Charney model, there is no temperature gradient at the upper boundary, but the temperature gradient at the lower boundary has the opposite sign as the interior pseudo-potential vorticity gradient, $\hat{\beta}$, so once again the necessary condition for instability is satisfied.

The Charney-Stern theorem may be interpreted as the requirement that at least two, counter-propagating (relative to the background flow) trains of Rossby waves must be supported by the fluid flow in order the normal mode instability to occur. One can also derive a Fjørtoft condition by taking the real part of (20.9); this shows, as in the Rayleigh instability problem, that the background flow must be configured so that the counter-propagating wave trains can become phase locked.