Notes on pressure coordinates

Robert Lindsay Korty

October 1, 2002

Obviously, it makes no difference whether the quasi-geostrophic equations are phrased in height coordinates (where x, y, z, t are the independent variables) or in pressure coordinates (where x, y, p, t are the independent variables); the systems of equations express the same physical meaning in both cases, and no information may be deduced from the equations phrased in pressure coordinates which could not also have been identified with the equations phrased in height coordinates. So in this sense, either set of variables is useful. The mathematics (i.e., the road taken to solve the equations), however, is not identical because the form that the equations take will differ between the two sets. The pressure coordinate system is preferable in most cases in the atmosphere, owing to the simpler form of the equations (principal among these simplifications is the extraction of density [which varies] from the equations of motion). In the ocean, or in any fluid in which an incompressibility assumption can be made, the equations are generally equally simple in either height or pressure coordinates. A major difference, in this regard, between the ocean and the atmosphere is that density is nearly constant in the ocean, but decays exponentially with altitude in the atmosphere. (As an aside, the pressure and density in the atmosphere drop by an order of magnitude roughly every 16 km; the surface pressure is about 1000 mb [100,000 Pa], at 16 km the pressure is only 100 mb [10,000 Pa], and by 48 km, the pressure is only 1 mb [100 Pa]. If the atmosphere were defined to be 100 km thick, the upper half by volume of a column of air contains only one percent of the mass of that column.)

Here, I'd like to address some benefits of the pressure coordinate system. First, the continuity equation in pressure coordinates takes a very simple form. Note that in height coordinates,

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \cdot (\rho \overrightarrow{u}) + \frac{\partial (\rho w)}{\partial z} = 0,$$

where ∇_z is the horizontal gradient operator in height coordinates $\left(\left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial}{\partial y}\right)_z\right)$; the subscript z is a notational reminder that these derivatives are taken holding *height* constant. In a fluid in which density varies, that fact must be taken into account in this equation. However, in pressure coordinates, the above expression may be rewritten as

$$\nabla_p \cdot \overrightarrow{u} + \frac{\partial \omega}{\partial p} = 0,$$

where ∇_p is the horizontal gradient operator in pressure coordinates $\left(\left(\frac{\partial}{\partial x}\right)_p + \left(\frac{\partial}{\partial y}\right)_p\right)$, and ω is the vertical velocity in pressure coordinates. This is a much simpler form of the same equation; the simplicity arises from the fact that the coordinate p represents the mass per unit area of the fluid above the corresponding isobaric surface (i.e., since a column of air has a surface pressure of roughly 1000 mb, half of the mass contained in it lies above the

500 mb surface). In the ocean, mass varies nearly linearly with depth; in the atmosphere, it decays exponentially with height (but, of course, varies linearly with pressure). From the continuity equation in pressure coordinates, interpreting p as a vertical coordinate and ω as a vertical velocity shows that in x, y, p-space, air moves as an incompressible fluid. The mathematical treatment of the equations of motion in pressure coordinates is considerably simplified for the atmosphere by using pressure coordinates. Aside from boundary conditions, it is no more difficult to use pressure coordinates in the ocean than it is to use height coordinates, but there is no benefit either. Since the ocean surface has variable pressure, height coordinates generally offer simpler boundary conditions in the ocean. Several texts (e.g., Gill, Holton) derive the pressure coordinate form of the continuity equation from first principles. Attached to this handout, I have shown how the height coordinate form may be mathematically transformed into the pressure coordinate form.

In the pressure coordinate form, the equations prove simpler than in the height coordinate form for the atmosphere. Some of the equations become formally identical to the equations for a homogeneous and incompressible fluid. Density drops out of the horizontal equations of motion, and the continuity equation is greatly simplified.

One trade off of pressure coordinates is that boundary conditions become more complicated. The surface of the Earth is not one of constant pressure; other coordinates, such as σ coordinates are sometimes used in general circulation models ($\sigma \equiv \frac{p}{p_s}$). But perhaps a bigger loss that results from this jump to pressure coordinates is that it is somewhat less intuitive to think in terms of pressure as a vertical coordinate. As noted above, though, the same information deduced in height coordinates may be found in pressure coordinates. As a simple example, consider the following.

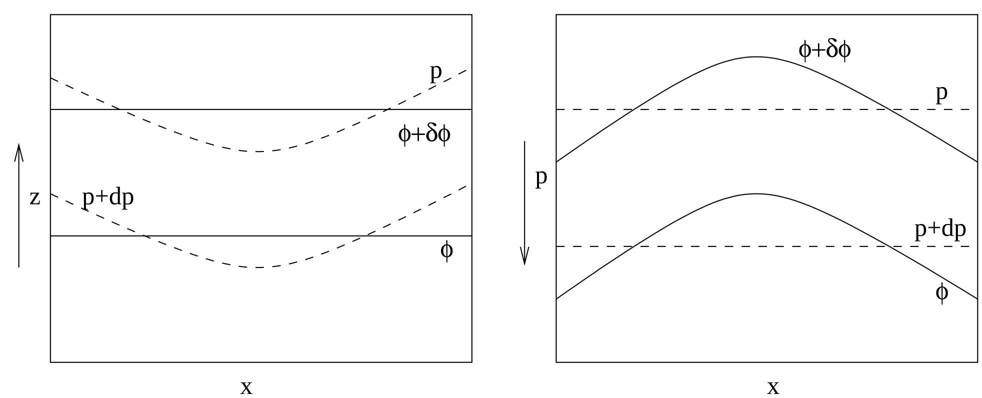

Figure 1: left panel: curves of constant pressure on height surfaces; right panel: the same picture showing how height varies along isobaric surfaces

If you were in an airplane confined to fly at a constant height of 5700 meters, your barometer would read somewhere near 500 mb. If you were flying along one of the constant ϕ lines depicted in the left panel by solid lines (remember, ϕ is simply gz) you would observe the pressure to go from high to low to high as you travel from left to right; it may go from 502 to 498 and back to 502 mb. However, if you instead confined your aircraft to fly at an altitude so as to maintain constant pressure (say, 500 mb), you would have to adjust your altitude. Your flight path is now along the dashed surfaces of constant pressure. You must go from a high altitude, to a lower one, and then rise again to maintain a constant pressure. You can visualize your motion with the left panel; you can see contours of your altitude with the right one. Note that as you travel from left to right, the value of ϕ at first decreases along your path, and then starts to rise half way through. Thus, whereas we study pressure perturbations (or, equivalently, ψ) in height coordinates, we study height perturbations (or, equivalently, ϕ) in pressure coordinates.

Note from the definition of S in equation (1), that S is **always positive**. Potential temperature increases with height, so it *decreases* with pressure (i.e., moving down in the atmosphere toward the surface). Thus, $d\theta/dp$ is always negative, and the minus sign contained in the definition makes S always positive. The relationship between N^2 and S is remarkably simple.

$$
\begin{aligned}
N^2 &= \frac{g}{\theta}\frac{d\theta}{dz} \\
&= \frac{g}{\theta}\frac{d\theta}{dp}(-\rho g) \text{ (hydrostatic approx.)} \\
&= -\frac{\rho^2 g^2}{\rho\theta}\frac{d\theta}{dp} \\
&= \rho^2 g^2 \left(\frac{-1}{\pi}\frac{d\theta}{dp}\right) \\
&= \rho^2 g^2 S \\
&= \frac{g^2}{\alpha^2} S \text{ (if you prefer specific volume to density)}
\end{aligned}
$$

So like N^2, S is always positive ($N^2 < 0$ is, like $S < 0$, a condition for convective instability). The minus sign contained in the definition of S is simply due to the difference in the direction that the vertical coordinate points in height and pressure coordinates. Both N^2 and S have the same physical meaning. The powers of ρg converting S to N^2 are due to the hydrostatic approximation which relates dp to dz.

I hope this helps clarify pressure coordinates. Just remember that a principal simplification comes from absorbing density variations into the vertical coordinate. Terms in the height coordinate form of the definition of q_p which look daunting when ρ is not constant:

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\rho\frac{\partial\psi}{\partial z}\right)$$

become simpler in pressure coordinates:

$$\frac{\partial}{\partial p}\left(\frac{f_o^2}{S}\frac{\partial\phi}{\partial p}\right).$$

A nice paper that goes over the transformation from height to pressure coordinates for all of the equations of motion was published in 1949 by Eliassen. I have a copy if any of you are interested.

Transforming the continuity equation into pressure coordinates

Robert Lindsay Korty
October 2, 2002

A number of texts cover the derivation of the continuity equation in pressure coordinates from first principles. Treatments of this may be found in Wallace and Hobbs (pp. 397-399), Holton (pp. 59-61), or Gill (pp. 181-182). Here the derivation is mathematical: the continuity equation is transformed from a system in which height is an independent variable (x, y, z, t) to one in which pressure is the vertical coordinate (x, y, p, t).

First, note the following coordinate transformations for an arbitrary variable quantity, A:

$$\begin{aligned}
\left(\frac{\partial A}{\partial x}\right)_p &= \left(\frac{\partial A}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial A}{\partial z} \\
\Rightarrow \left(\frac{\partial A}{\partial x}\right)_z &= \left(\frac{\partial A}{\partial x}\right)_p - \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial A}{\partial z} \\
&= \left(\frac{\partial A}{\partial x}\right)_p - \frac{1}{g}\left(\frac{\partial \phi}{\partial x}\right)_p \frac{\partial A}{\partial z}.
\end{aligned}$$

And, with the hydrostatic approximation, $\delta p = -\rho g \delta z \Rightarrow \frac{1}{\delta z} = -\frac{\rho g}{\delta p}$ (in pressure coordinates, $\frac{\partial \phi}{\partial p} = -\frac{1}{\rho}$), we may write expressions for the transformation of derivatives:

$$\left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_p + \rho\left(\frac{\partial \phi}{\partial x}\right)_p \frac{\partial A}{\partial p}. \tag{2}$$

Similarly, derivatives with respect to y or t are transformed:

$$\left(\frac{\partial A}{\partial y}\right)_z = \left(\frac{\partial A}{\partial y}\right)_p + \rho\left(\frac{\partial \phi}{\partial y}\right)_p \frac{\partial A}{\partial p} \tag{3}$$

$$\left(\frac{\partial A}{\partial t}\right)_z = \left(\frac{\partial A}{\partial t}\right)_p + \rho\left(\frac{\partial \phi}{\partial t}\right)_p \frac{\partial A}{\partial p}. \tag{4}$$

Before transforming the continuity equation from height to pressure coordinates, make an Eulerian expansion of the material derivative in the definition of w. This will be needed to convert w to ω. From the definition of w,

$$\mathrm{w} \equiv \frac{dz}{dt} = \left(\frac{\partial z}{\partial t}\right)_p + u\left(\frac{\partial z}{\partial x}\right)_p + v\left(\frac{\partial z}{\partial y}\right)_p + \omega\frac{\partial z}{\partial p}.$$

Using the definition of ϕ, and the hydrostatic approximation to replace $\frac{\partial z}{\partial p}$,

$$\mathrm{w} = \frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)_p + \frac{u}{g}\left(\frac{\partial \phi}{\partial x}\right)_p + \frac{v}{g}\left(\frac{\partial \phi}{\partial y}\right)_p - \frac{\omega}{\rho g}. \tag{5}$$

(As an aside, if numerical accuracy is not important, ω may be approximated by $-\rho g\mathrm{w}$ to within about five percent. Note that ω and w have opposite signs. Rising motion corresponds to a decrease in pressure, which gives a negative ω.)

The continuity equation, expressed in height coordinates, may be written:

$$\underbrace{\left(\frac{\partial \rho}{\partial t}\right)_z}_{A} + \underbrace{\left(\frac{\partial \rho u}{\partial x}\right)_z}_{B} + \underbrace{\left(\frac{\partial \rho v}{\partial y}\right)_z}_{C} + \underbrace{\left(\frac{\partial \rho \mathrm{w}}{\partial z}\right)_{x,y,t}}_{D} = 0 \tag{6}$$

We now must transform the derivatives holding z constant into derivatives which hold p constant. By equation (2), term B in equation (6) is

$$\begin{aligned}\left(\frac{\partial \rho u}{\partial x}\right)_z &= \left(\frac{\partial \rho u}{\partial x}\right)_p + \rho\left(\frac{\partial \phi}{\partial x}\right)_p \frac{\partial \rho u}{\partial p} \\ &= \rho\left(\frac{\partial u}{\partial x}\right)_p + u\left(\frac{\partial \rho}{\partial x}\right)_p + \rho\frac{\partial}{\partial p}\left[\rho u\left(\frac{\partial \phi}{\partial x}\right)_p\right].\end{aligned}$$

But $\left(\frac{\partial \rho}{\partial x}\right)_p = 0$: $\left(\frac{\partial}{\partial x}\right)_p\left(-\frac{1}{g}\frac{\partial p}{\partial z}\right) = -\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right)_p = 0$, so term B of equation (6) is simply

$$\left(\frac{\partial \rho u}{\partial x}\right)_z = \rho\left(\frac{\partial u}{\partial x}\right)_p + \rho\frac{\partial}{\partial p}\left[\rho u\left(\frac{\partial \phi}{\partial x}\right)_p\right] \tag{7}$$

and from equations (3) and (4) we may write terms C and A from equation (6) as

$$\left(\frac{\partial \rho v}{\partial y}\right)_z = \rho\left(\frac{\partial v}{\partial y}\right)_p + \rho\frac{\partial}{\partial p}\left[\rho v\left(\frac{\partial \phi}{\partial y}\right)_p\right] \tag{8}$$

$$\begin{aligned}\left(\frac{\partial \rho}{\partial t}\right)_z &= \left(\frac{\partial \rho}{\partial t}\right)_p + \rho\left(\frac{\partial \phi}{\partial t}\right)_p \frac{\partial \rho}{\partial p} \\ &= \left(\frac{\partial}{\partial t}\right)_p\left[-\frac{1}{g}\frac{\partial p}{\partial z}\right] + \rho\frac{\partial}{\partial p}\left[\rho\left(\frac{\partial \phi}{\partial t}\right)_p\right] \\ &= -\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right)_p + \rho\frac{\partial}{\partial p}\left[\rho\left(\frac{\partial \phi}{\partial t}\right)_p\right]\end{aligned}$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t}\right)_z = \rho\frac{\partial}{\partial p}\left[\rho\left(\frac{\partial \phi}{\partial t}\right)_p\right] \tag{9}$$

Using the hydrostatic approximation, we may rewrite term D in equation (6) as

$$\frac{\partial \rho \mathrm{w}}{\partial z} = -\rho g\frac{\partial \rho \mathrm{w}}{\partial p} \tag{10}$$

Now, plugging (7), (8), (9), and (10) into equation (6) yields:

$$\rho\frac{\partial}{\partial p}\left[\rho\left(\frac{\partial\phi}{\partial t}\right)_p+\rho u\left(\frac{\partial\phi}{\partial x}\right)_p+\rho v\left(\frac{\partial\phi}{\partial y}\right)_p\right]+\rho\left(\frac{\partial u}{\partial x}\right)_p+\rho\left(\frac{\partial v}{\partial y}\right)_p-\rho g\frac{\partial\rho\mathrm{w}}{\partial p}=0$$

And from (5), we note that the term inside the square brackets in the expression above is equal to $\omega+\rho g\mathrm{w}$. Making this substitution results in the continuity equation transformed into pressure coordinates.

$$\begin{aligned}\rho\frac{\partial}{\partial p}\left[\omega+\rho g\mathrm{w}\right]+\rho\left(\frac{\partial u}{\partial x}\right)_p+\rho\left(\frac{\partial v}{\partial y}\right)_p-\rho g\frac{\partial\rho\mathrm{w}}{\partial p} &= 0\\ \left(\frac{\partial u}{\partial x}\right)_p+\left(\frac{\partial v}{\partial y}\right)_p+\frac{\partial\omega}{\partial p} &= 0\\ \nabla_p\cdot\overrightarrow{u}+\frac{\partial\omega}{\partial p} &= 0 \qquad (11)\end{aligned}$$

Note that the continuity equation no longer includes any explicit reference to density. This is a principal advantage of pressure coordinates. For the ocean, or for any fluid for which an incompressibility assumption can be made, the continuity equation in height coordinates reduces to $\nabla_z\cdot\overrightarrow{u}+\frac{\partial\mathrm{w}}{\partial z}=0$, such that height or pressure coordinates are equally simple to use. In the atmosphere, equation (11) is a much simpler form than the height coordinate form (equation (6)) since an incompressibility assumption can not be made; in pressure coordinates, the exponential variation of density is subsumed in the coordinate transformation and air moves like an incompressible fluid in x, y, p-space. Mathematically, the pressure coordinate form of the continuity equation for the atmosphere has the same form as the height coordinate form of the continuity equation for the ocean.