Tropical Cyclone Inner Core Dynamics

Assumptions

- Axisymmetric flow
- Gradient and hydrostatic balance above PBL
- Troposphere neutral to slantwise moist convection outside eye
- Moist adiabatic lapse rate in eye above inversion
- Well developed anticyclone at storm top

Local energy balance (from previous lecture):

$$\frac{M}{r_b^2} \cong -\left(T_b - T_o\right)\frac{ds^*}{dM}$$

Definitions:

$$\frac{f}{2}R^2 \equiv M = rV + \frac{f}{2}r^2 \qquad \text{"potential radius"}$$

$$\chi^* \equiv \left(T_s - T_o\right)\left(s^* - s_a^*\right)$$

$$\chi \equiv \left(T_s - T_o\right)\left(s - s_a\right) \qquad \left(s_a = s_a^*\right)$$

$$\chi_s \equiv \left(T_s - T_o\right)\left(s_{0a}^* - s_a\right) = \text{constant}$$

Scaling:

$$\chi^*, \chi \to \chi_s\left(\chi^*, \chi\right)$$

$$R, r \to \frac{\sqrt{\chi_s}}{f}\left(R, r\right)$$

Scaled equations:

$$\frac{1}{r^2} = -\frac{2}{R^3}\frac{\partial \chi^*}{\partial R},$$

$$R^2 = 2rV + r^2 \simeq 2rV \qquad \text{(core)}$$

$$\to \boxed{V^2 \simeq -\frac{R}{2}\frac{\partial \chi^*}{\partial R}} \qquad (1)$$

Conservation of angular momentum (dimensional):

$$\frac{dM}{dt} = -gr\frac{\partial \tau_\theta}{\partial p}$$

Integrate over depth of PBL:

$$\Delta p_b \frac{dM}{dt} = -gr\tau_{\theta s} \cong -gr\rho_s C_D V^2$$

$$\cong -gr\frac{p_s}{R_d T_s} C_D V^2$$

Scaling for time:

$$t \rightarrow C_D^{-1} \frac{R_d T_s}{g p_s} \Delta p_b \chi_s^{-1/2} t$$

Nondimensional angular momentum equation:

$$\frac{dR}{dt} = -r \frac{V^2}{R}$$

$$\text{But} \quad r \simeq \frac{R^2}{2V}$$

$$\rightarrow \boxed{\frac{dR}{dt} \simeq -\frac{1}{2} RV} \qquad (2)$$

Nondimensional PBL entropy equation:

$$\frac{d\chi}{dt} = \frac{C_k}{C_D} V\left(\chi_0^* - \chi\right) + \varepsilon V^3 - F_b, \qquad (3)$$

$$\varepsilon \equiv \frac{T_s - T_o}{T_s}$$

Time derivative in R space:

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} + \dot{R}\frac{\partial}{\partial R} + \omega\frac{\partial}{\partial P}$$

Assume X well mixed in boundary layer,

use $\dot{R} = -\frac{1}{2}RV$:

Also assume that F_b balances source terms outside of eyewall:

$$\frac{\partial \chi}{\partial \tau} = \begin{cases} \frac{1}{2}RV\frac{\partial \chi}{\partial R} + \frac{C_k}{C_D}V\left(\chi_0^* - \chi\right) + \varepsilon V^3 & \textit{eyewall} \\ 0 & \textit{elsewhere} \end{cases}$$

$$\mathit{But} \quad V^2 \simeq -\frac{R}{2}\frac{\partial \chi^*}{\partial R} = -\frac{R}{2}\frac{\partial \chi}{\partial R} \quad \mathit{in\ eyewall}$$

$$\rightarrow \frac{\partial \chi}{\partial \tau} = \beta\left[\frac{C_k}{C_D}V\left(\chi_0^* - \chi\right) - \left(1-\varepsilon\right)V^3\right], \quad (4)$$

$$\beta = 1 \quad \mathit{in\ eyewall}$$

$$\beta = 0 \quad \mathit{elsewhere}$$

(Steady state solution:)

$$V^2 = \frac{1}{1-\varepsilon}\frac{C_k}{C_D}\left(\chi_0^* - \chi\right)$$

Differentiate (4) with respect to *R* and use $V^2 \simeq -\frac{R}{2}\frac{\partial \chi}{\partial R}$:

$$\frac{\partial V}{\partial \tau} = \frac{\beta}{4}\frac{R}{V}\frac{\partial V}{\partial R}\left[3(1-\varepsilon)V^2 - \frac{C_k}{C_D}\left(\chi_0^* - \chi\right)\right]$$

$$-\frac{C_k}{C_D}\frac{\beta}{2}V^2$$

$$+\frac{R}{4}\frac{\partial \beta}{\partial R}\left[(1-\varepsilon)V^2 - \frac{C_k}{C_D}\left(\chi_0^* - \chi\right)\right]$$

First term: Propagation; Second term: damping;

Third term: Amplification

Note that first term steepens V gradient when $V^2 > V_{max}^2$

$$\frac{\partial \beta}{\partial R} < 0 \quad \textit{necessary for amplification}$$

V gradient cannot steepen indefinitely:

$$\varsigma = \frac{V}{r} + \frac{\partial V}{\partial r},$$

$$\frac{\partial}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial}{\partial R} = \frac{r}{R}\left(1 + \frac{V}{r} + \frac{\partial V}{\partial r}\right)\frac{\partial}{\partial R} = \frac{r}{R}\left(1+\varsigma\right)\frac{\partial}{\partial R}$$

$$\rightarrow \varsigma = \frac{V}{r} + \frac{r}{R}\left(1+\varsigma\right)\frac{\partial V}{\partial R}$$

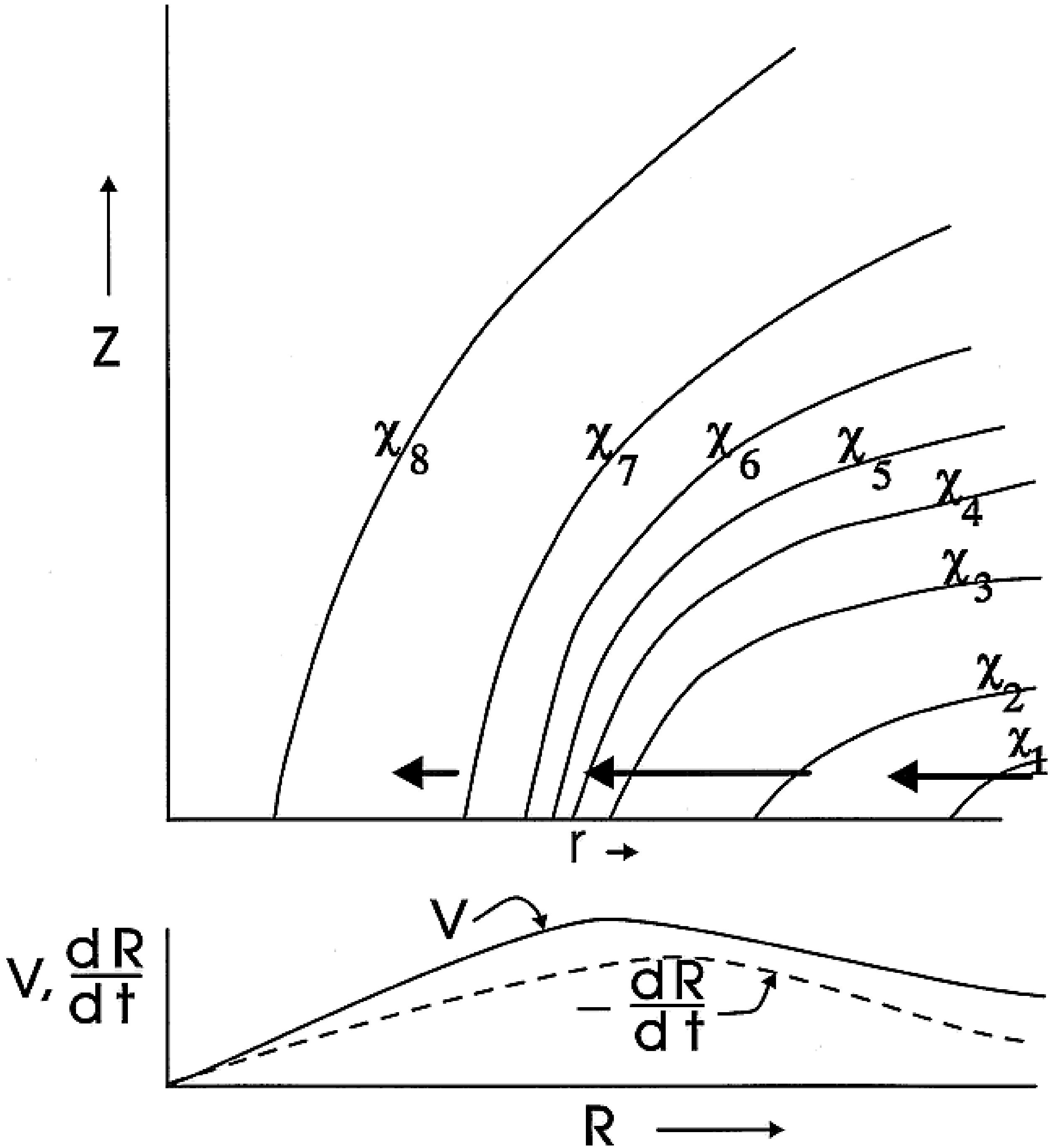
$$\varsigma = \frac{\frac{V}{r} + \frac{r}{R}\frac{\partial V}{\partial R}}{1 - \frac{r}{R}\frac{\partial V}{\partial R}}$$

$$\varsigma \to \infty \quad when \quad \frac{\partial V}{\partial R} \to \frac{R}{r} \cong \frac{2V}{R}$$

Eyewall undergoes frontal collapse!

This can only be prevented by 3-D eddies

$$\dot{R} \simeq -\frac{1}{2} R V$$



Simplified amplification model:

$$\frac{\partial \chi}{\partial \tau} = \beta \left[\frac{C_k}{C_D} V\left(\chi_0^* - \chi\right) - \left(1-\varepsilon\right) V^3 \right],$$

$$\beta = 1 \quad in\ eyewall$$

$$\beta = 0 \quad elsewhere$$

$$\chi_0^* = 1 - AP = 1 + A\left(\chi + \frac{1}{2}V^2\right)$$

$$V^2 = -\frac{R}{2}\frac{\partial \chi^*}{\partial R}$$

$$Enforce \quad \chi \leq \chi^*$$

How to handle frontal collapse in model?

Three methods:

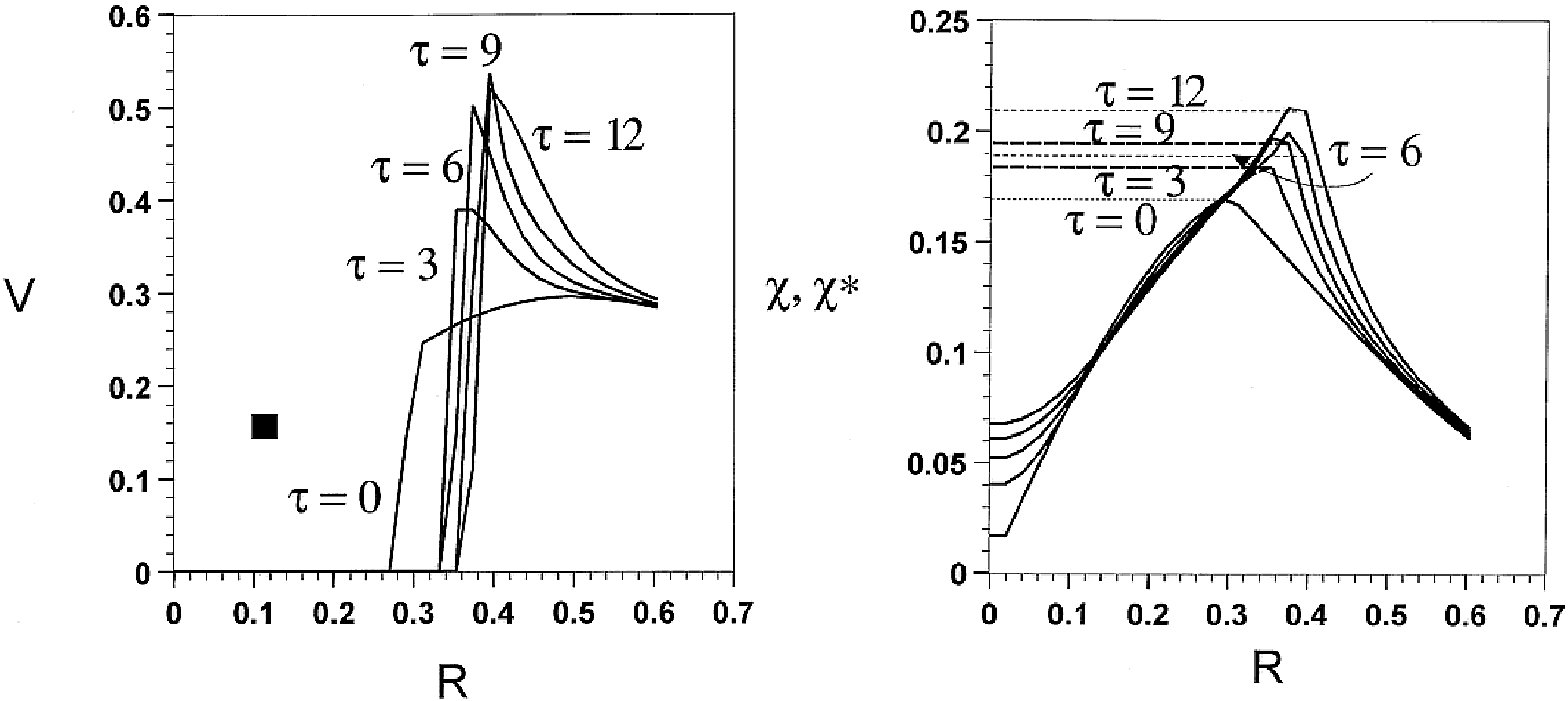
1: Zero diffusion model:

For $r < r_m$:

$$V = 0$$

$$\chi = \chi^* = constant$$

Does not prevent frontal collapse



V
$\tau = 9$
$\tau = 12$
$\tau = 6$
$\tau = 3$
$\tau = 0$
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
R
χ, χ^*
$\tau = 12$
$\tau = 9$
$\tau = 6$
$\tau = 3$
$\tau = 0$
0.25
0.2
0.15
0.1
0.05
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
R

2. Minimum diffusion model: Just enough radial diffusion to prevent failure of coordinate transformation.

From expression for vertical component of vorticity, we enforce

$$\frac{\partial V}{\partial R} \leq \frac{2V}{R}$$

Inside the outermost radius, R_{crit}, where this is violated, we take

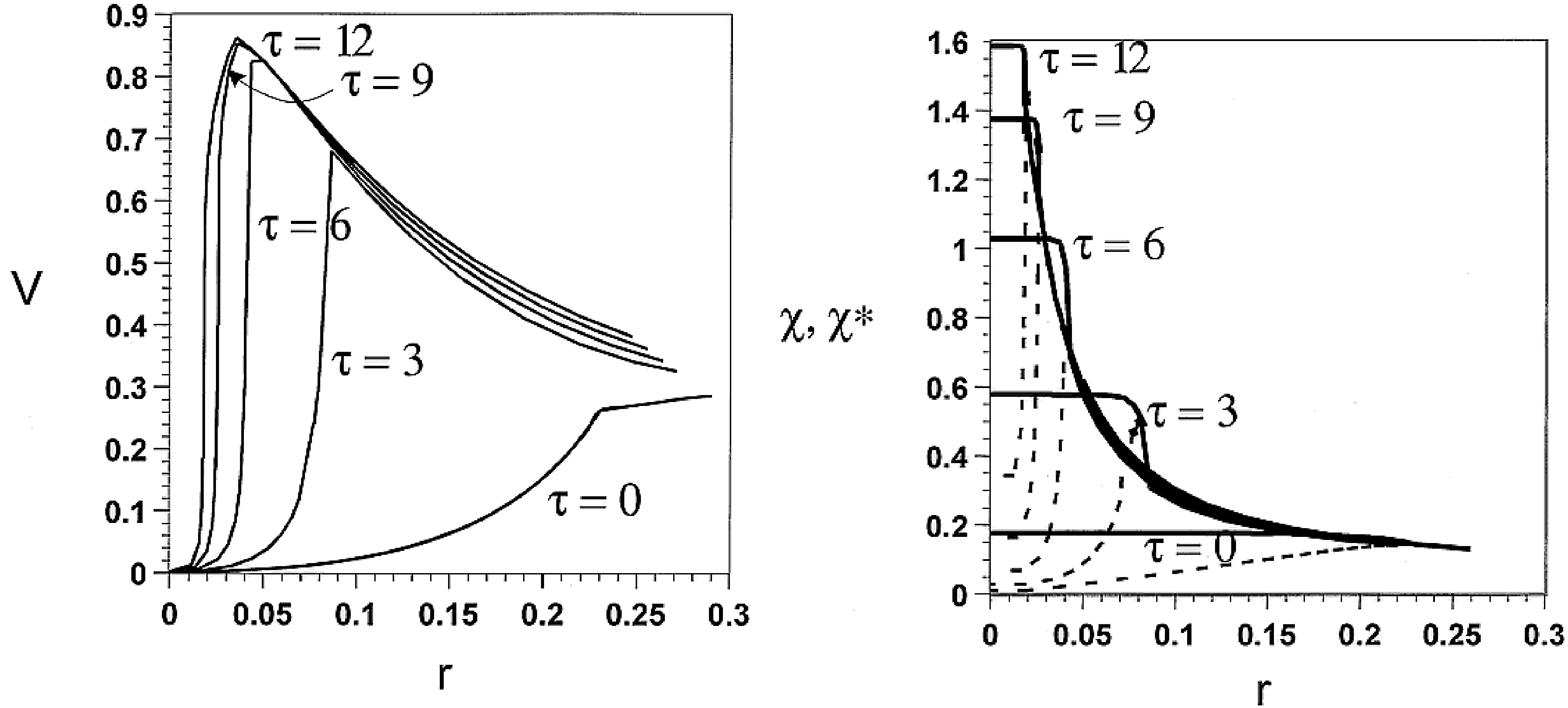
$$\frac{\partial V}{\partial R} = \frac{2V}{R}$$

$$\rightarrow V = V_{crit}\left(\frac{R}{R_{crit}}\right)^2$$

By integration of $$\frac{\partial \chi^*}{\partial R} = -\frac{2V^2}{R}$$

$$\chi^* = \chi_{crit} + \frac{1}{2}V_{crit}^2\left(1 - \left(\frac{R}{R_{crit}}\right)^4\right) \quad for\ R < R_{crit},$$

$$\chi^* = \chi \qquad for\ R \geq R_{crit}$$



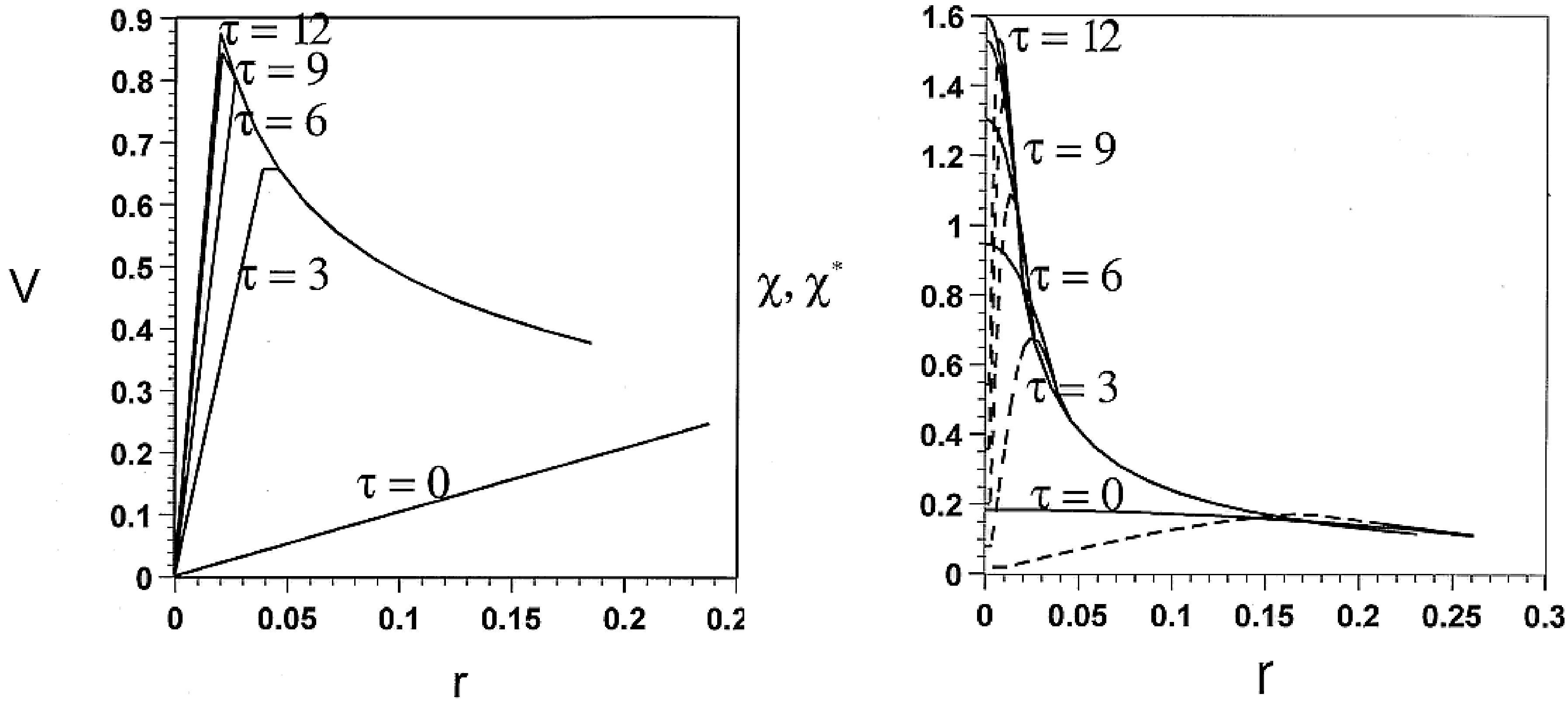
V
r
τ = 12
τ = 9
τ = 6
τ = 3
τ = 0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.05
0.1
0.15
0.2
0.25
0.3
χ, χ*
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2

3. Maximum diffusion model: 3-D turbulence perfectly efficient in establishing constant angular velocity inside r_m:

$$V = V_{max}\frac{r}{r_m} = V_{max}\frac{R}{R_m}$$

$$\chi^* = \chi_m + V_{max}^2\left[1 - \left(\frac{R}{R_m}\right)^2\right] \quad for\ R < R_m,$$

$$\chi^* = \chi \qquad for\ R \geq R_m$$



V
r
τ = 12
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0.9
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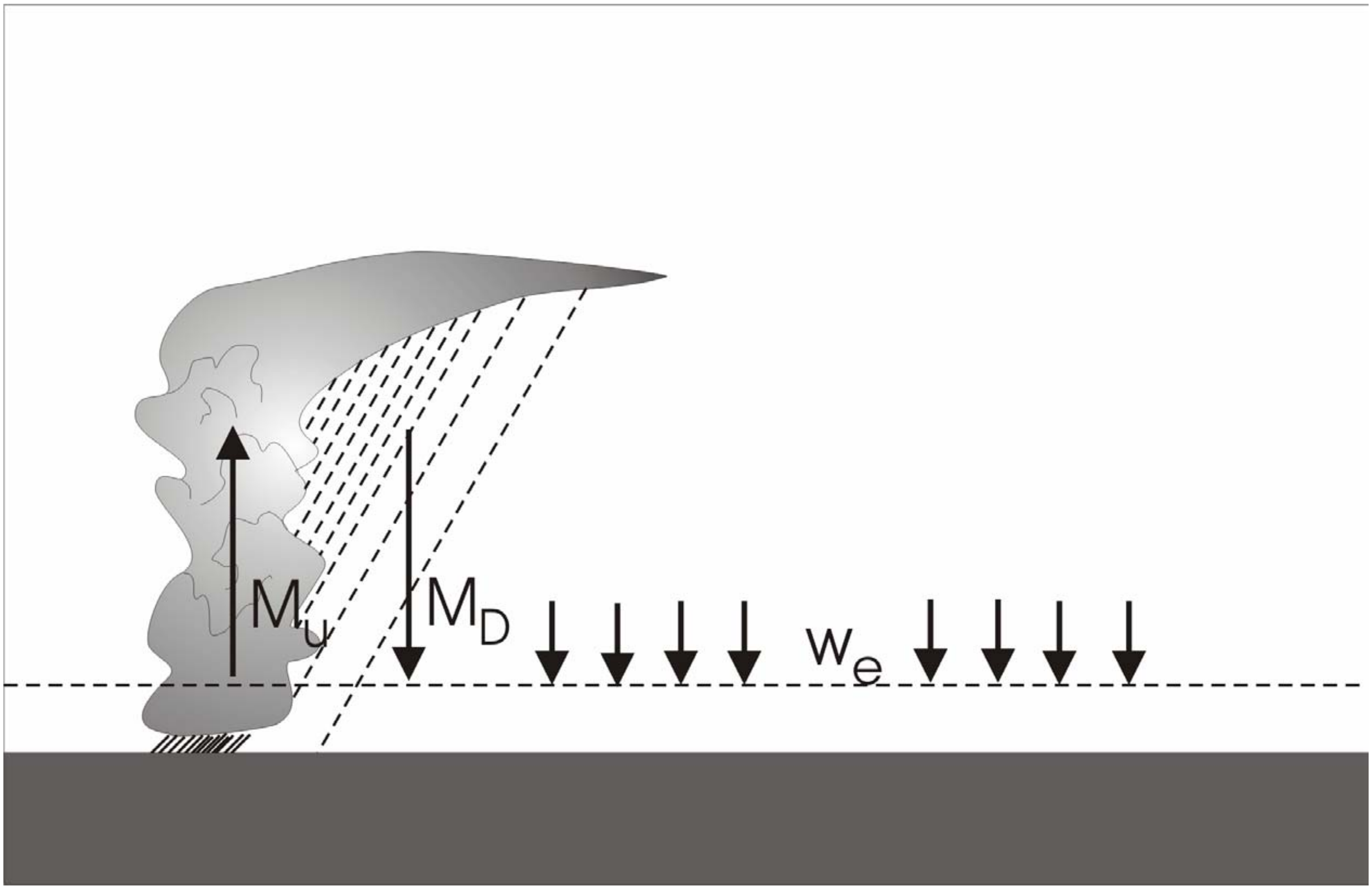
0.4
0.3
0.2
0.1
0
0.05
0.1
0.15
0.2
χ, χ*
r
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0.25
0.3

Tropical Cyclone Structure

- Hydrostatic and gradient balance above PBL everywhere
- Moist adiabatic lapse rates on angular momentum surfaces above PBL everywhere
- Boundary layer quasi-equilibrium (with horizontal entropy advection and dissipative heating) in rain area
- Radiative-subsidence balance in outer region

Boundary layer quasi-equilibrium



Assume that boundary layer entropy changes slowly in angular momentum space:

$$\frac{ds}{dt} = \frac{\partial s}{\partial \tau} + \frac{dM}{dt}\frac{\partial s}{\partial M}$$

$$hT_s \frac{\partial s}{\partial \tau} = C_k \,|\mathbf{V}|\left(k_s^{\ *} - k\right) + C_D \,|\mathbf{V}|^3 - \left(M_u - w\right)\left(h_b - h_m\right) - hT_s \frac{dM}{dt}\frac{\partial s}{\partial M}.$$

PBL angular momentum: $$h\frac{d\overline{M}}{dt} = -C_D r\,|\mathbf{V}|\,V$$

$$hT_s \frac{\partial s}{\partial \tau} = C_k \,|\mathbf{V}|\left(k_s^{\ *} - k\right) + C_D \,|\mathbf{V}|^3 - \left(M_u - w\right)\left(h_b - h_m\right) + T_s C_D r\,|\mathbf{V}|\,V \frac{\partial s}{\partial M}. \quad (1)$$

In rain area, $M_u > 0, \qquad s = s^*:$

Use thermal wind balance, neglecting $\frac{1}{r_o^2}$:

$$T_s C_D r\,|\mathbf{V}|\,V \frac{\partial s}{\partial M} \simeq -C_D\,|\mathbf{V}|\,V \frac{T_s}{T_s - T_o}\frac{M}{r}.$$

Use in (1), approximating $\frac{M}{r}$ *by* V *and* $|\mathbf{V}|$ *by* V:

$$M_u = w + \frac{1}{h_b - h_m}\left[C_k V\left(k_s^* - k\right) - C_D V^3 \frac{T_o}{T_s - T_o}\right]. \qquad (2)$$

Closure for convective updraft mass flux.

Free tropopshere thermodynamic balance:

$$\frac{\partial s^*}{\partial \tau} = \frac{\Gamma_d}{\Gamma_m}\left[\left(M_u - M_d - w\right)\frac{\partial s_d}{\partial z} + \frac{\dot{Q}_{rad}}{T}\right] \qquad (3)$$

Representation of downdrafts:

$$M_d = (1-\varepsilon)M_u$$

Steady state version of (3):

$$w = -w_{rad} + \varepsilon M_u, \qquad (4)$$

$$w_{rad} \equiv -\dot{Q}_{rad} \Big/ T\frac{\partial s_d}{\partial z}$$

Outside rain area,

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r} = -w_{rad} \qquad M_{u=0} \tag{5}$$

Within rain area, substitute (2) into (4):

$$(1-\varepsilon)\frac{1}{r}\frac{\partial \psi}{\partial r} = -w_{rad} + \frac{\varepsilon}{h_b - h_m}\left(C_k V \left(k_s^{\ *} - k \right) - C_D V^3 \frac{T_o}{T_s - T_o} \right) \quad M_u > 0. \tag{6}$$

System closed using PBL angular momentum balance:

$$\psi \frac{\partial M}{\partial r} = C_D r^2 V^2,$$

or, equivalently,

$$\frac{\partial(rV)}{\partial r} = \frac{C_D r^2 V^2}{\psi} - fr.$$

Apply scaling:

$$V \rightarrow v_{\max} V,$$

$$r \rightarrow \frac{v_{\max}}{f} r,$$

$$\psi \rightarrow C_D \frac{v_{\max}^3}{f^2} \psi,$$

$$w_{rad} \rightarrow C_D v_{\max} w_Q,$$

$$M_u \rightarrow C_D v_{\max} M_u$$

Nondimensional system:

$$\frac{\partial\left(rV\right)}{\partial r}=\frac{r^2V^2}{\psi}-r, \tag{7}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial r}=-w_Q \qquad for\ M_u=0, \tag{8}$$

$$\frac{1-\varepsilon}{r}\frac{\partial\psi}{\partial r}=-w_Q+\varepsilon\Lambda\left(V-V^3\right) \quad for\ M_u>0, \tag{9}$$

$$M_u=\frac{1}{1-\varepsilon}\left[-w_Q+\Lambda\left(V-V^3\right)\right] \tag{10}$$

$$\Lambda\equiv\frac{C_k}{C_D}\frac{k_s{}^*-k}{h_b-h_m}$$

Boundary condition: $\psi = 0 \; at \; r = r_0$

From (7): $V = 0 \; at \; r = r_0$

Simple case: $\Lambda, \varepsilon, w_Q$ constant

Outside rain area, solution to (8) is then

$$\psi = \frac{1}{2} w_Q \left(r_o^2 - r^2 \right)$$

Substitute into (7):

$$\frac{\partial \left(rV \right)}{\partial r} = \frac{2r^2 V^2}{w_Q \left(r_o^2 - r^2 \right)} - r$$

For $w_Q << 1$, dominant balance is

$$V^2 \approx \frac{1}{2} w_Q \frac{r_o^2 - r^2}{r}$$

For $r^2 << r_0^2$,

$$V \approx r^{-\frac{1}{2}}$$

In rain area, assume that $\varepsilon \Lambda V >> w_Q$ $V^2 << 1$

$$\frac{\partial \psi}{\partial r} \simeq \frac{\varepsilon \Lambda}{1-\varepsilon} rV, \tag{11}$$

$$\frac{\partial}{\partial r}\left(rV\right) \simeq \frac{r^2 V^2}{\psi}. \tag{12}$$

(11) and (12) have power law solution:

$$V \approx r^{-n}$$

$$n \equiv \frac{\varepsilon\Lambda - 2(1-\varepsilon)}{\varepsilon\Lambda - (1-\varepsilon)}.$$

Realistic only if

$$\Lambda > 2\frac{1-\varepsilon}{\varepsilon}.$$

(In numerical solutions to be presented, $n = 2/3$)

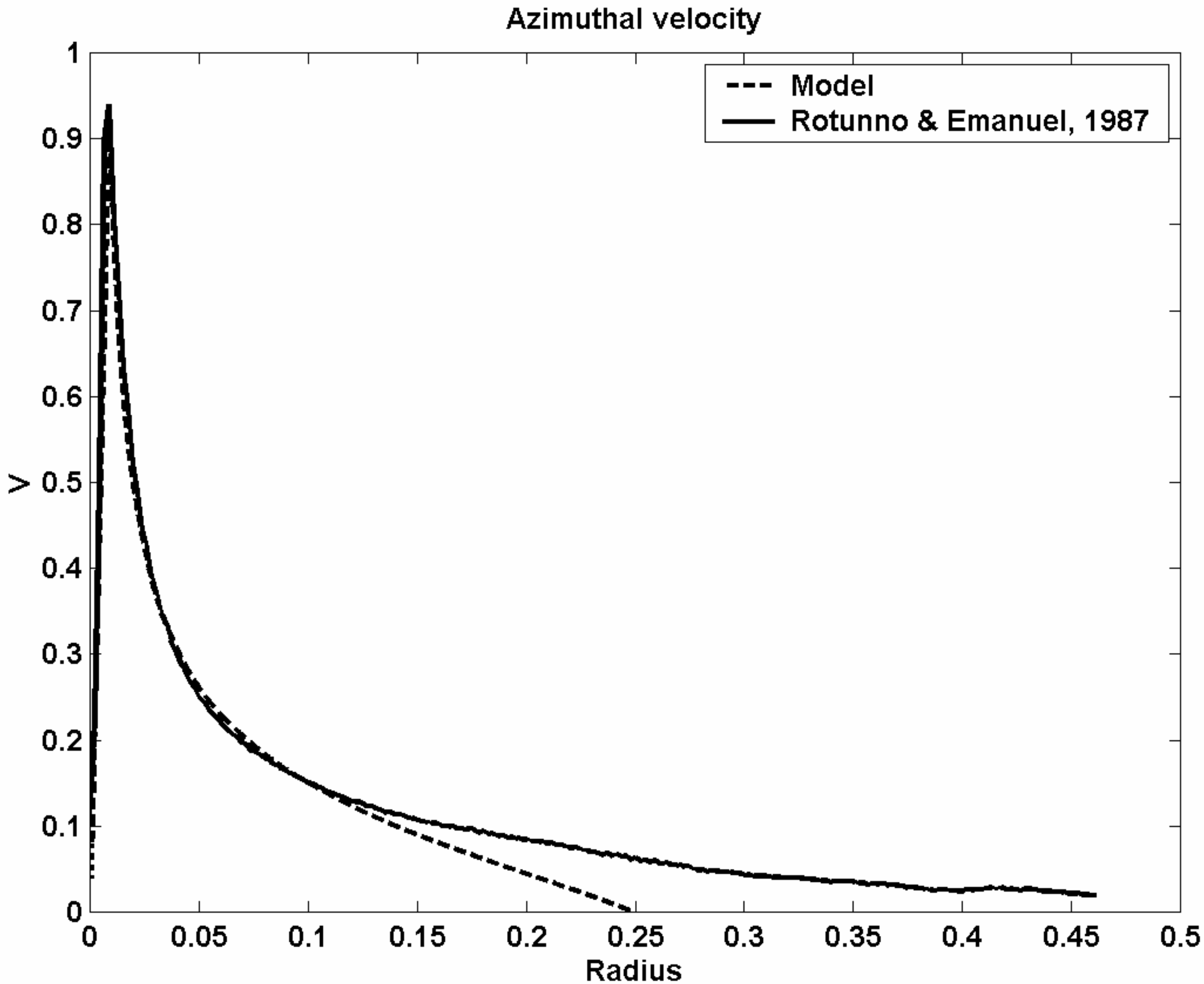
Numerical solution of (7)-(10): March inward from $r = r_0$:

$\Lambda = 1$

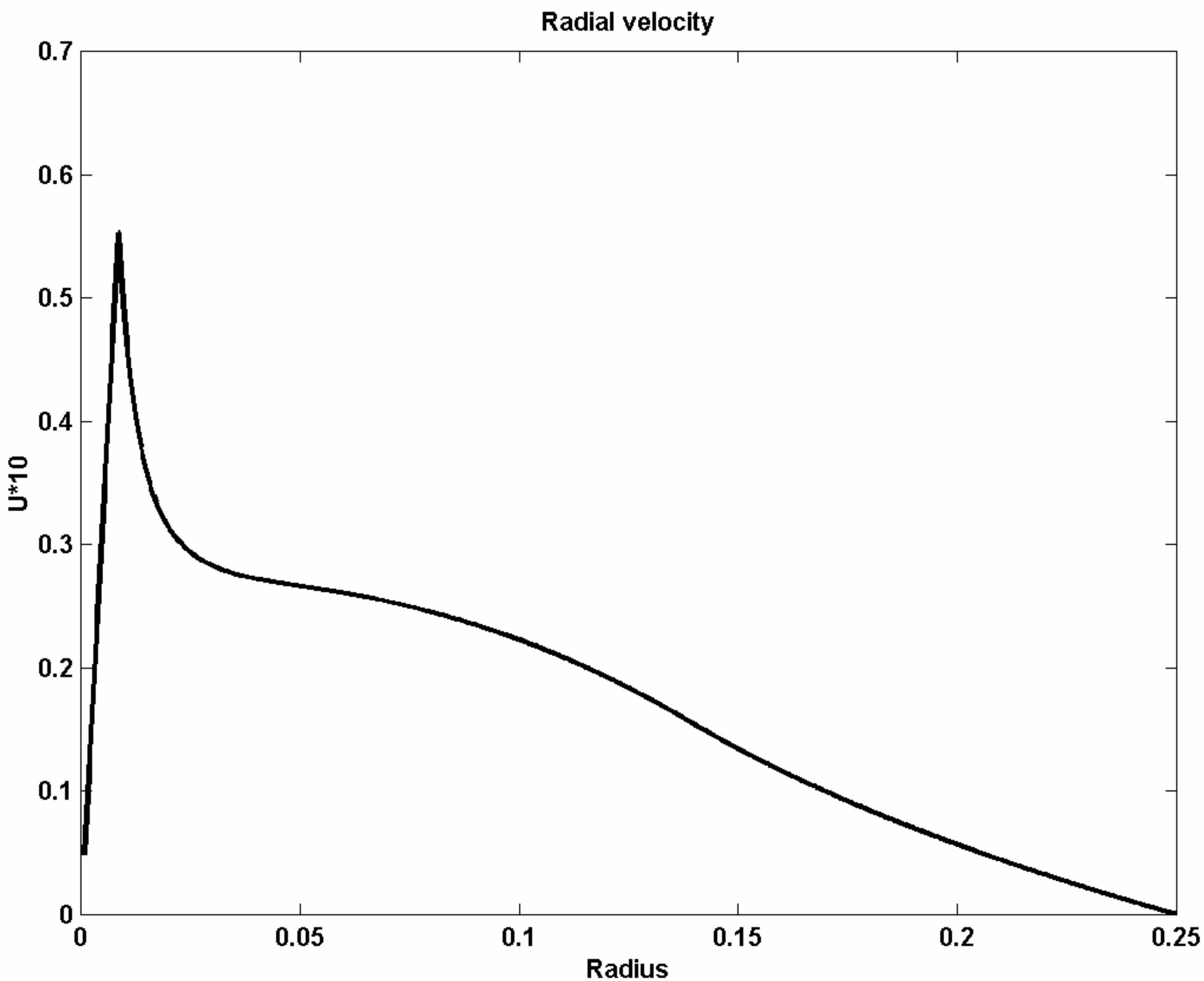
$\varepsilon = 0.8$

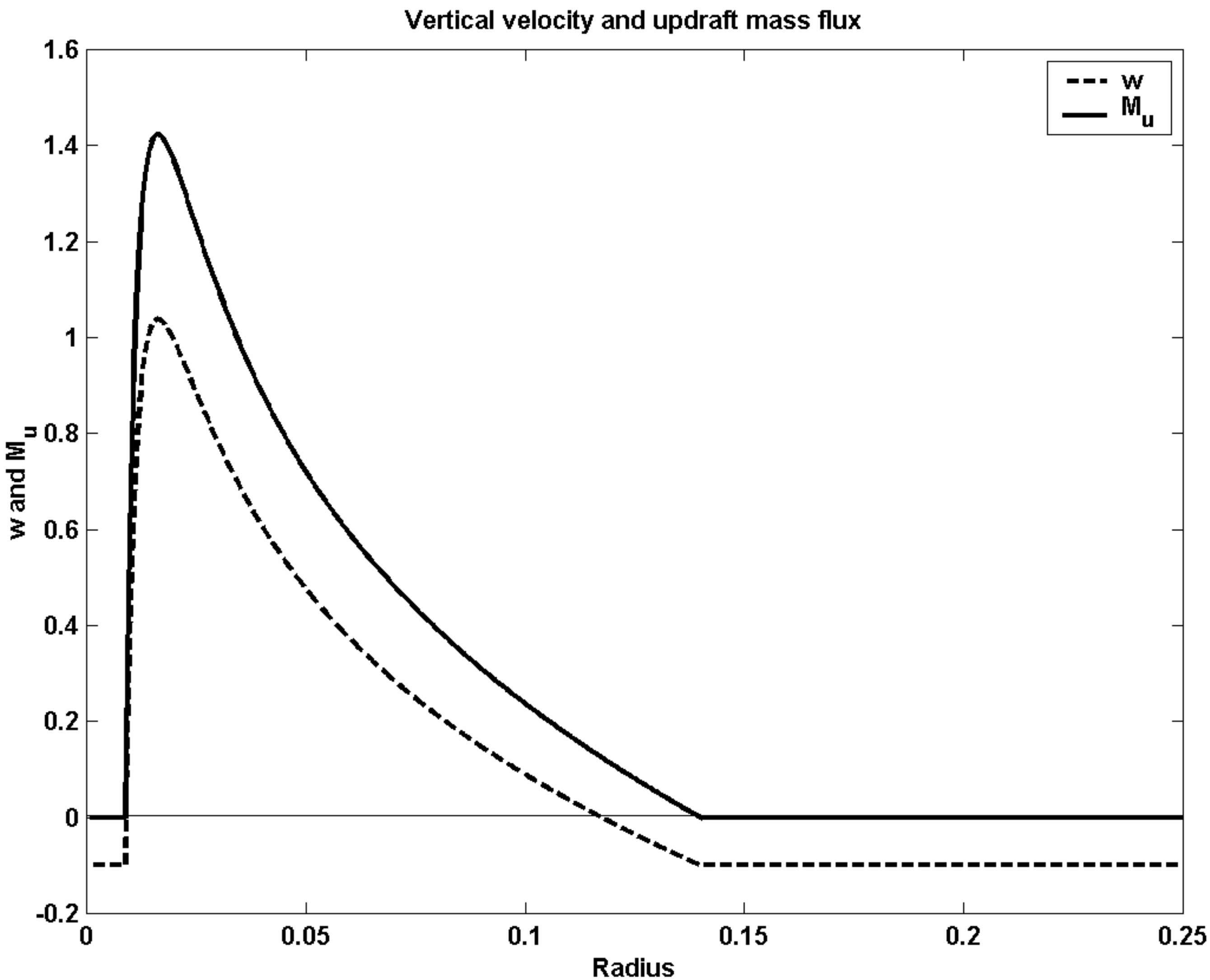
$w_Q = 0.1$

$r_o = 0.25$



Note: longer tail in R&E model owing to different formulation of $\dot{Q}$

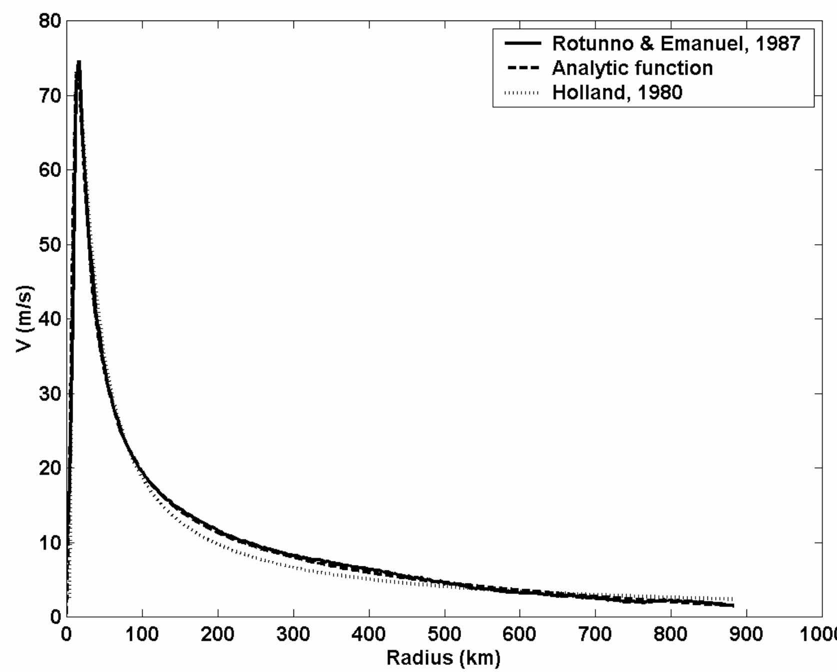

Radial velocity
U*10
Radius
0
0.05
0.1
0.15
0.2
0.25
0.1
0.2
0.3
0.4
0.5
0.6
0.7

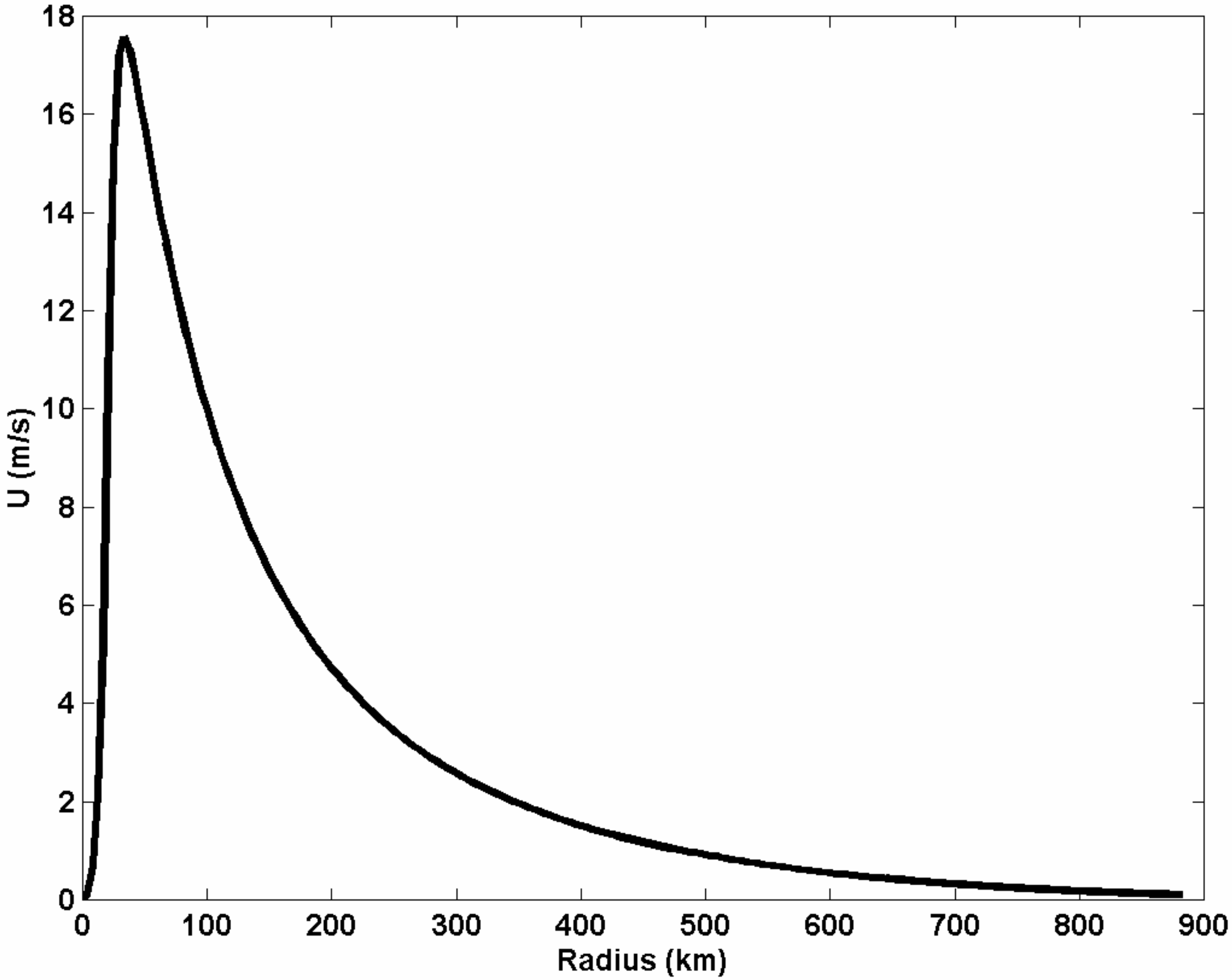


Vertical velocity and updraft mass flux
w
M_u
w and M_u
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.05
0.1
0.15
0.2
0.25
Radius

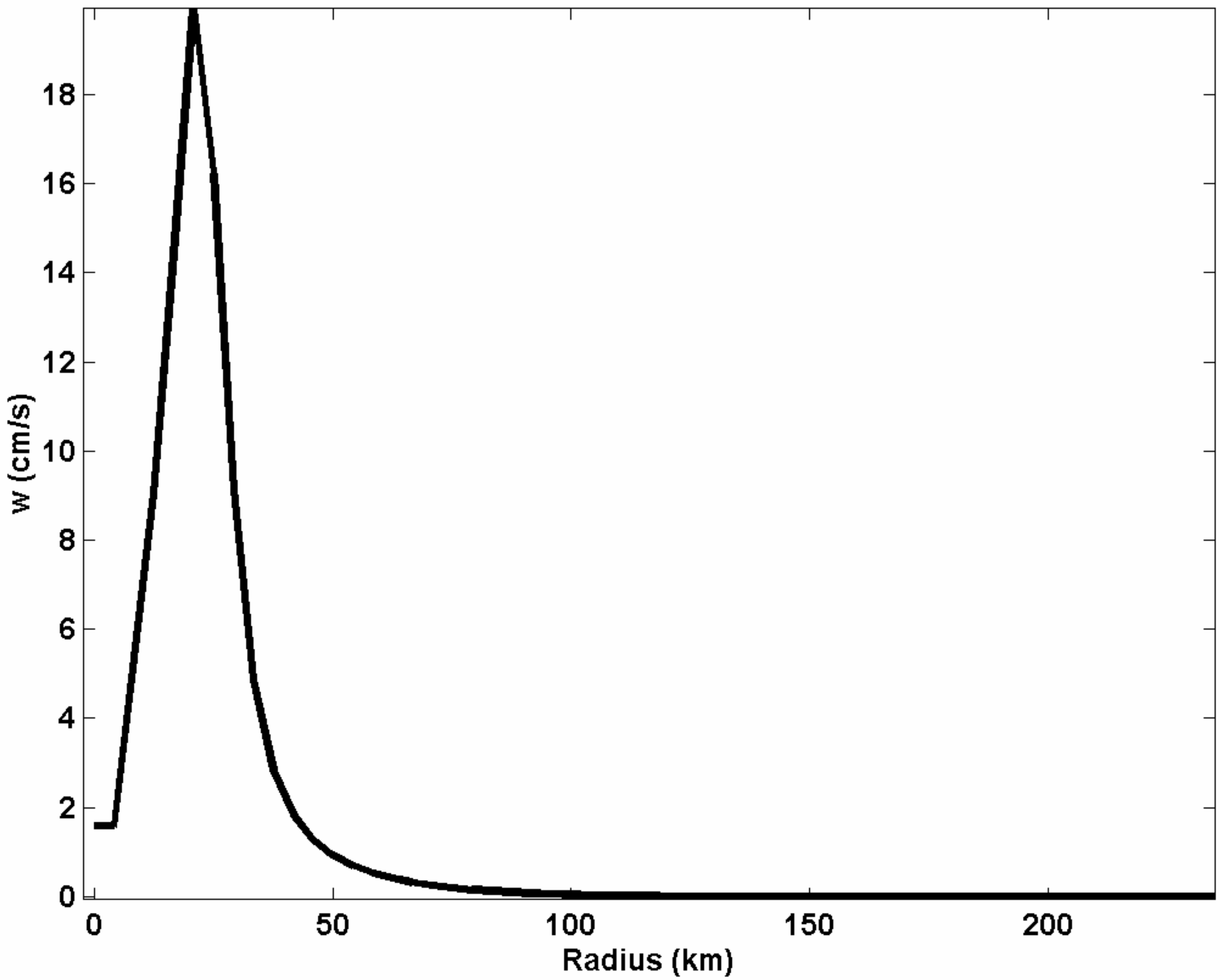
Curve fit to numerical solutions:

$$V^2 = V_m^2 \left(\frac{r_0 - r}{r_0 - r_m} \right)^2 \left(\frac{r}{r_m} \right)^{2m} \left[\frac{(1-b)(n+m)}{n + m\left(\frac{r}{r_m} \right)^{2(n+m)}} + \frac{b(1+2m)}{1 + 2m\left(\frac{r}{r_m} \right)^{2m+1}} \right],$$

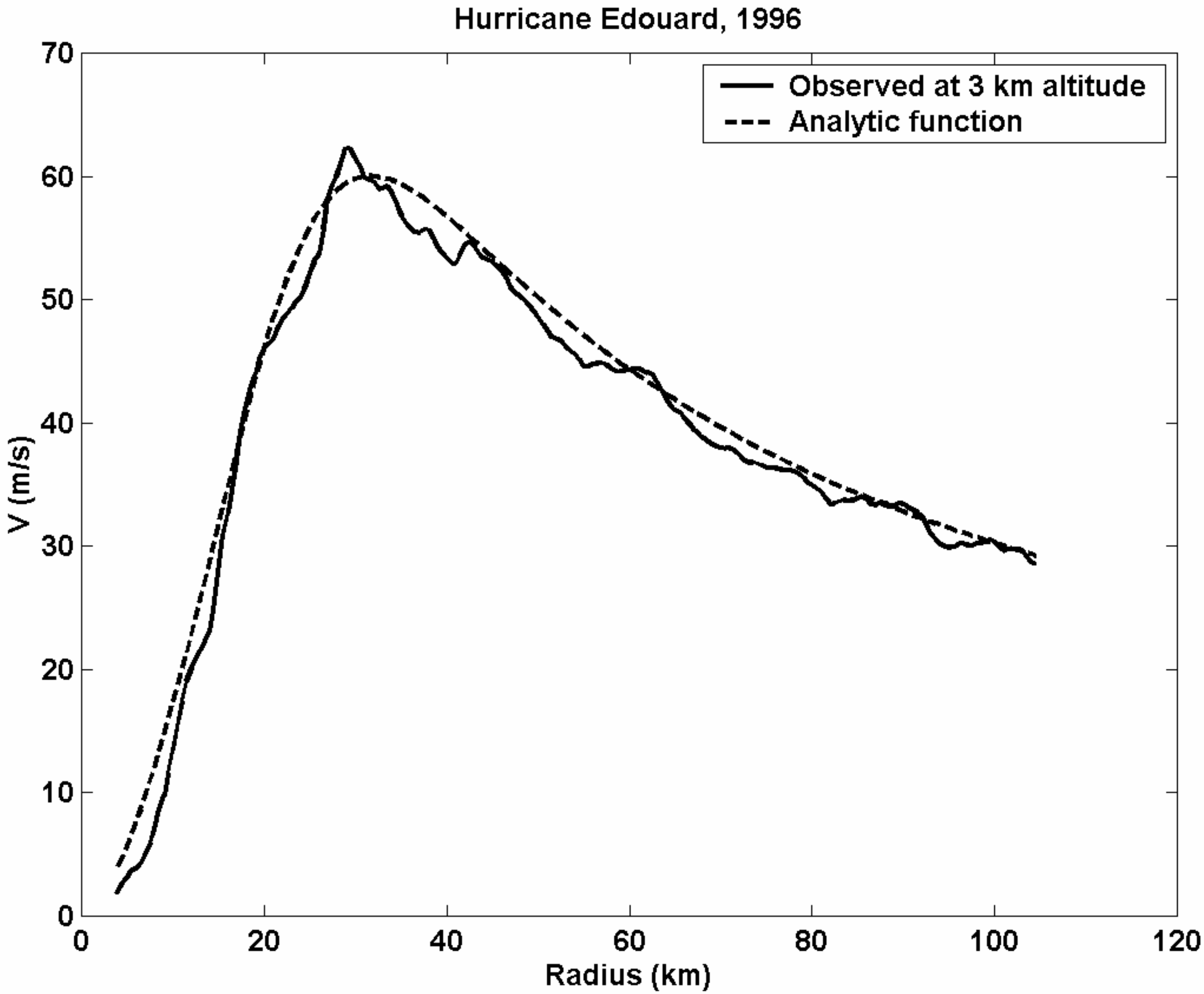




U (m/s)
0
2
4
6
8
10
12
14
16
18
0
100
200
300
400
500
600
700
800
900
Radius (km)



w (cm/s)
Radius (km)
0
2
4
6
8
10
12
14
16
18
0
50
100
150
200


Hurricane Edouard, 1996
Observed at 3 km altitude
Analytic function
V (m/s)
Radius (km)
0
10
20
30
40
50
60
70
0
20
40
60
80
100
120