Time-dependent, axisymmetric model phrased in R space

- Hydrostatic and gradient balance above PBL
- Moist adiabatic lapse rates on M surfaces above PBL
- Boundary layer quasi-equilibrium
- Deformation-based radial diffusion
Potential Radius:

\[
\frac{f}{2} R^2 \equiv M = rV + \frac{f}{2} r^2
\]  

(1)

Local energy conservation:

\[
\frac{1}{r_b^2} - \frac{1}{r_t^2} = -2 \frac{(T_s - T_t)}{f^2 R^3} \frac{\partial s^*}{\partial R}
\]

(2)

Differentiate in time:

\[
\frac{1}{r_b^3} \frac{\partial r_b}{\partial \tau} - \frac{1}{r_t^3} \frac{\partial r_t}{\partial \tau} = \frac{(T_s - T_t)}{f^2 R^3} \frac{\partial}{\partial R} \frac{\partial s^*}{\partial \tau}
\]

(3)
Mass continuity:

\[ ru = \frac{\partial \psi}{\partial p} , \]

\[ r \omega = - \frac{\partial \psi}{\partial r} \]

Transform to potential radius coordinates:

\[ ru = r \frac{dr}{dt} = r \left[ \frac{\partial r}{\partial \tau} + \frac{\partial r}{\partial R} \frac{dR}{dt} + \frac{\partial r}{\partial P} \frac{dP}{dt} \right] = \frac{\partial \psi}{\partial p} , \]

\[ \rightarrow r \frac{\partial r}{\partial \tau} = \frac{\partial \psi}{\partial p} - r \omega \frac{\partial r}{\partial P} = \frac{\partial \psi}{\partial p} + \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial P} = \frac{\partial \psi}{\partial P} \]
\[
\frac{\partial r^2}{\partial \tau} = 2 \frac{\partial \psi}{\partial P}
\]

Define \( \psi_0 \) as streamfunction at top of boundary layer and use simple finite difference in vertical:

\[
\frac{\partial r_b^2}{\partial \tau} \approx 2(\psi_0 - \psi),
\]

\[
\frac{\partial r_t^2}{\partial \tau} \approx 2\psi \tag{4}
\]
PBL flow:

\[
\frac{\partial \mathbf{r}}{\partial \tau} + r \frac{\partial \mathbf{r}}{\partial R} \frac{dR}{dt} = \frac{\partial \psi}{\partial P}
\]

Angular momentum balance:

\[
\frac{f}{2} \frac{dR^2}{dt} = -gr \frac{\partial \tau_\theta}{\partial P}
\]

\[
\rightarrow r \frac{\partial r^2}{\partial R^2} \left( \frac{2}{f} \right) g \frac{\partial \tau_\theta}{\partial P} \approx -\frac{\partial \psi}{\partial P},
\]

\[
\psi_0 = -\frac{2}{f} g \frac{\partial r^2}{\partial R^2} \tau_s = \frac{2}{f} g \frac{\partial r^2}{\partial R^2} \rho_s C_D \left| \mathbf{V} \right| V
\]

\[
V = \frac{f}{2} \frac{R^2 - r^2}{r}
\]
Saturation entropy:

\[
\frac{\partial S^*}{\partial \tau} = \frac{\Gamma_d}{\Gamma_m} \left[ \left( M_u - M_d - w \right) \frac{\partial S_d}{\partial z} + \frac{\dot{Q}_{\text{rad}}}{T} \right]
\]

Downdraft:

\[
M_d = \left( 1 - \varepsilon_p \right) M_u
\]

Boundary layer entropy:

\[
h_s \frac{\partial S}{\partial \tau} = C_k | \mathbf{V} | (s_0^* - s) + C_D | \mathbf{V} |^3 - (M_u - w_0)(s - s_m) + C_D r | \mathbf{V} | V \frac{\partial s}{fR}\partial R
\]

Used to define \( M_{ueq} \) when \( > 0 \); otherwise, equation integrated for \( s \)
Relaxation equation:

\[
\frac{\partial M_u}{\partial \tau} = \frac{M_{ueq} - M_u}{\tau_c}
\]

Precipitation efficiency:

\[
\mathcal{E}_p = \frac{S_m - S_{m0}}{S - S_{m0}}
\]

Middle troposphere entropy:

\[
\frac{\partial S_m}{\partial \tau} = \Lambda M_u (s - s_m) + \dot{Q}_{rad}
\]
Radiation: \[ \dot{Q}_{rad} \approx -(s^* - s^*(t = 0)) \]

Radial diffusion added to equations for \( r_b, r_t, s^*, \text{and} \ s_m \)

\[
D_b = -\frac{1}{R} \frac{\partial}{\partial R} \left[ r_b^2 \nu_b \frac{\partial}{\partial r_b} \left( \frac{R^2}{r_b^2} \right) \right]
\]

\[
D_t = -\frac{1}{R} \frac{\partial}{\partial R} \left[ r_t^2 \nu_t \frac{\partial}{\partial r_t} \left( \frac{R^2}{r_t^2} \right) \right]
\]

\[
D_{s^*} = \frac{\partial}{\partial r_b^2} \left( r_b \nu_b \frac{\partial s^*}{\partial r_b} \right) \quad D_{s_m} = \frac{\partial}{\partial r_m^2} \left( r_m \nu_b \frac{\partial s_m}{\partial r_m} \right)
\]
\[ v_i = l^2 \left| r_i \frac{\partial}{\partial r_i} \left( \frac{R^2}{r_i^2} \right) \right| \]

Note that surface saturation entropy depends on pressure, which is calculated from gradient wind balance using \( V \)

Complete equations summarized in Emanuel (1995), posted on course web page.
Model behavior

- **Control**
- **Theory**
- **Weak initial vortex**

The graph shows the maximum surface wind speed (m/s) over time (days). The blue line represents the control scenario, the green dotted line represents the theoretical model, and the red line indicates a weak initial vortex scenario.
Saturate troposphere inside 100 km in initial state:
Character of control simulation
Cumulus mass flux, from 0 to 18.1277 m/s
Azimuthal velocity, from -0.0423 to 66.4187 m/s
Radial velocity, from -27.7593 to 74.5129 m/s
Vertical velocity, from -0.2099 to 19.6568 m/s (- values X 10)
Equivalent potential temperature, from 329.8344 to 368.7422 K

Radius (km)

z (km)