Dynamics of the Zonal-Mean, Time-Mean Tropical Circulation
First consider a hypothetical planet like Earth, but with no continents and no seasons and for which the only friction acting on the atmosphere is at the surface.

This planet has an exact nonlinear equilibrium solution for the flow of the atmosphere, characterized by

1. Every column is in radiative-convective equilibrium,

2. Wind vanishes at planet’s surface

3. Horizontal pressure gradients balanced by Coriolis accelerations
Hydrostatic balance:

\[
\frac{\partial p}{\partial z} = -\rho g
\]

In pressure coordinates:

\[
\frac{\partial \phi}{\partial p} = -\alpha,
\]

where

\[
\alpha = \frac{1}{\rho} \equiv \text{specific volume},
\]

\[
\phi = gz \equiv \text{geopotential}
\]
Geostrophic balance on a sphere:

\( \left( \frac{\partial \phi}{a \partial \theta} \right)_p = -2\Omega \sin \theta u_{rel} - \frac{u_{rel}^2}{a} \tan \theta \)

Can be phrased in terms of absolute angular momentum per unit mass:

\[ M = a \cos \theta (\Omega a \cos \theta + u) \]

\[ \rightarrow \quad \frac{\partial \phi}{\partial \theta} = -\sin \theta \left[ \frac{M^2 - \Omega^2 a^4 \cos^4 \theta}{a^2 \cos^3 \theta} \right] \]
Eliminate $\phi$ using hydrostatic equation:

$$\frac{1}{a^2} \frac{\tan \theta}{\cos^2 \theta} \frac{\partial M^2}{\partial p} = \left( \frac{\partial \alpha}{\partial \theta} \right)_p = \left( \frac{\partial T}{\partial p} \right)_{s^*} \left( \frac{\partial s^*}{\partial \theta} \right)_p$$

(Thermal wind equation)

Take $s^* = s^*(\theta)$  (convective neutrality)

Integrate upward from surface ($p=p_0$), taking $u=0$ at surface:
\[ M^2 = a^2 \cos^2 \theta \left[ \Omega^2 a^2 \cos^2 \theta - \cot \theta (T_s - T) \frac{\partial s^*}{\partial \theta} \right] \]

\[ \rightarrow u^2 + 2\Omega a \cos \theta u + \cot \theta (T_s - T) \frac{\partial s^*}{\partial \theta} = 0. \]

\[ u \equiv -\frac{(T_s - T) \frac{\partial s^*}{\partial \theta}}{2\Omega \sin \theta \frac{\partial s^*}{\partial y}} \]

Implies strongest west-east winds where entropy gradient is strongest, weighted toward equator.
Two potential problems with this solution:

1. Not enough angular momentum available for required west-east wind,

2. Equilibrium solution may be unstable
How much angular momentum is needed?

At the tropopause,

\[ M^2 = a^2 \cos^2 \theta \left[ \Omega^2 a^2 \cos^2 \theta - \cot \theta (T_s - T_t) \frac{\partial s^*}{\partial \theta} \right] \]

How much angular momentum is available?

\[ M^2_{eq} = \Omega^2 a^4 \]

\[ \rightarrow (T_s - T_t) \frac{\partial s^*}{\partial \theta} \geq - \frac{\Omega^2 a^2 \tan \theta \left(1 - \cos^4 \theta\right)}{\cos^2 \theta} \]
More generally, we require that angular momentum decrease away from the equator:

\[
\frac{1}{\sin \theta} \frac{\partial M^2}{\partial \theta} < 0
\]

\[
\rightarrow 4\Omega^2 a^2 \cos^3 \theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \cos^2 \theta \cot \theta (T_s - T_t) \frac{\partial s^*}{\partial \theta} \right] > 0.
\]

Let \( y \equiv \sec^2 \theta \)

\[
\rightarrow 1 + y^3 \frac{\partial}{\partial y} \left( \frac{T_s - T_t}{\Omega^2 a^2} \frac{\partial s^*}{\partial y} \right) > 0
\]
What happens when this condition is violated?

Overturning circulation must develop

Acts to drive actual entropy distribution back toward criticality

Postulate constant angular momentum at tropopause:

$$M_t = \Omega a^2$$
Integrate thermal wind equations down from tropopause:

\[
M^2 = \Omega^2 a^4 + a^2 \cos^2 \theta \cot \theta (T - T_t) \frac{\partial s^*}{\partial \theta}
\]

\[
\rightarrow u \approx \frac{\Omega a}{2} \left( \frac{1}{\cos^3 \theta} - \cos \theta \right) + \frac{T - T_t}{2\Omega a \sin \theta} \frac{\partial s^*}{\partial \theta}
\]

Note: \( u \) at surface always < 0 for supercritical entropy distribution
Critical entropy distribution:

\[
\frac{T_s - T_t}{\Omega^2 a^2} \frac{\partial s^*}{\partial y} = \frac{1}{2} \left( \frac{1}{y^2} - 1 \right)
\]

\[
\rightarrow \quad \frac{T_s - T_t}{\Omega^2 a^2} s^* = \frac{T_s - T_t}{\Omega^2 a^2} s_{eq}^* \left( 1 - \frac{1}{2} \left( y + \frac{1}{y} \right) \right)
\]

\[
= \frac{T_s - T_t}{\Omega^2 a^2} s_{eq}^* \left( 1 - \frac{1}{2} \left( \cos^2 \theta + \frac{1}{\cos^2 \theta} \right) \right)
\]
Violation results in large-scale overturning circulation, known as the Hadley Circulation, that transports heat poleward and drives surface entropy gradient back toward its critical value.
Figure 2: Critical (open circles) and observed (solid line) distributions of $\theta_{cs}$ for every month at the 600 mb level as a function of latitude
Simulations by Held and Hou (1980)
Zonal wind at tropopause
The cross-equatorial Hadley Circulation
Numerical simulation using a 2-D primitive equation model with parameterized convection (Pauluis, 2004)

Imposed sea surface temperature
Why is there ascent on cold side of equator?

Meridional pressure gradient above the PBL:

\[
\frac{\partial \varphi}{\partial y} = -\mathbf{V} \cdot \nabla v - 2\Omega \sin \theta u \\
\approx -\frac{1}{2} \frac{\partial v^2}{\partial y} - 2\Omega \sin \theta u
\]
Hydrostatic:

\[
\varphi_{PBL} = \varphi - R \frac{\Delta p_{pbl}}{p} (T_v)_{pbl}
\]

\[
\rightarrow \quad \frac{\partial \varphi_{pbl}}{\partial y} = \frac{\partial \varphi}{\partial y} - R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl}
\]

\[
= -\frac{1}{2} \frac{\partial v^2}{\partial y} - 2\Omega \sin \theta u - R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl}
\]
PBL momentum:

\[ V \cdot \nabla v - F_v = -\left( \frac{\partial \varphi}{\partial y} \right)_{pbl} - 2\Omega \sin \theta u_{pbl} \]

\[ \rightarrow \frac{1}{2} \left( \frac{\partial v^2}{\partial y} \right)_{pbl} - F_v \approx \frac{1}{2} \left( \frac{\partial v^2}{\partial y} \right) + R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl} + 2\Omega \sin \theta (u - u_{pbl}) \]

\[ F_v \approx -C_D \frac{|V|}{h} v \]

\[ F_v \text{ dominant when } h < \sim C_D \Delta y \]

\[ \left( C_D \approx 10^{-3} \right) \]
Thin, frictionally dominated PBL:

\[
C_D \frac{|V|}{h} v \approx R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl} + 2\Omega \sin \theta u
\]

\(u\) constrained by angular momentum conservation:

\[
u_{\text{min}} = \frac{\Omega a}{\cos \theta} \left[ \sin^2 \theta - \sin^2 \theta_0 \right],
\]

\[
u_{\text{max}} = \frac{\Omega a}{\cos \theta} \sin^2 \theta
\]
Cold side of equator:

\[ C_D \frac{|V|}{h} v \approx R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl} + 2\Omega^2 a \tan \theta \left[ \sin^2 \theta - \sin^2 \theta_0 \right] \]

Warm side of equator:

\[ C_D \frac{|V|}{h} v \approx R \frac{\Delta p_{pbl}}{p} \left( \frac{\partial T_v}{\partial y} \right)_{pbl} + 2\Omega^2 a \tan \theta \sin^2 \theta \]
maximum merdional wind

\[ v \text{ (m/s)} \]

\[ \text{latitude} \]
Angular momentum and stream function, $\Delta P = 50 \text{ mb}$

\[ \Delta p_{\text{pbl}} = 50 \text{ mb} \]
Angular momentum and stream function, $\Delta P = 100 \text{ mb}$

$$\Delta p_{pbl} = 100 \text{ mb}$$
Angular momentum and stream function, $\Delta P = 200\, mb$

\[ \Delta p_{pbl} = 200\, mb \]
January Zonal Mean OLR, Vertical and Meridional Wind, 1979-1993 from ECMWF

Contour interval 1 mm s$^{-1}$

Shading Red Positive (Upward)
July Zonal Mean OLR, Vertical and Meridional Wind, 1979-1993 from ECMWF

Contour interval 1 mm s$^{-1}$

Shading Red Positive (Upward)
Two-D primitive equation model

• Parameterizations of
  – convection
  – fractional cloudiness
  – radiation
  – surface fluxes

• Ocean mixed layer energy budget

• Model forced by annual cycle of solar radiation

• Available for class projects
Surface temperature (C) from 10.9485 to 30.9417
Precipitation (mm/day) from 0.0518 to 19.3306
Surface $v$ (m/s) from -11.9717 to 10.3175
Outgoing longwave radiation (W/m²) from 138.9862 to 319.5031
Angular Momentum and Streamfunction
Relative humidity from 0.2221 to 98.6573